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Generalization of Lie's counting theorems for second and third order ordinary differential equations

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GENERALIZATION OF LIE'S COUNTING THEOREMS
FOR SECOND AND THIRD ORDER
ORDINARY DIFFERENTIAL EQUATIONS

A Thesis

Presented to

the Graduate Faculty of the
University of the Pacific

In Partial Fulfillment
of the Requirements for the Degree of
Master of Science

by

Suzanne Marie Davison

August 1973

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INTRODUCTION

This work concerns two new theorems which count the maximum possible number of independent generators of a certain form which leave an ordinary differential equation of second or third order covariant. Sophus Lie has derived such theorems for a particular class of transformations. The new theorems contain Lie's theorems as a subcase, and are therefore called "generalized" theorems.

In Chapter I some background theory on Lie generators and their extension is given. Chapter II concerns some detailed mathematical background necessary for the understanding of the generalized theorems, including a discussion of the properties of Vandermonde determinants. The theorems for second order equations, both Lie's and the new one, are given in Chapter III along with some examples of equations which produce the maximum allowed number of generators. Chapter IV gives Lie's r^{th} ($r > 2$) order theorem, a generalization for third order cases and a discussion of a method of handling the general r^{th} ($r > 2$) case, again with examples. A general summary and conclusion is contained in Chapter V.

CHAPTER I: General Background Theory¹

Consider the group transformation law

$$\begin{aligned} x_1 &= \phi(x, y; a) \\ y_1 &= \psi(x, y; a) \end{aligned} \quad (1-1)$$

where ϕ and ψ are infinitely differentiable functions of x and y and are analytic in a group parameter a . The parameter can always be chosen such that $a=0$ corresponds to the identity, i.e.,

$$\begin{aligned} x_1 &= \phi(x, y; 0) = x \\ y_1 &= \psi(x, y; 0) = y \end{aligned} \quad (1-2)$$

Take $f(x, y)$ to be an infinitely differentiable function of x and y . The transformation (1-1) acting on the arguments of such a function has the effect

$$f(x, y) \rightarrow f(x_1, y_1) \stackrel{\text{def}}{=} f_1(x, y; a)$$

Assuming that f_1 is analytic in a and expanding in a Taylor series around the identity $a=0$ we obtain

$$f_1 = f + \left(\frac{\partial f_1}{\partial a} \right)_{a=0} a + \left(\frac{\partial^2 f_1}{\partial a^2} \right)_{a=0} \frac{a^2}{2!} + \dots$$

Now

$$\left(\frac{\partial f_1}{\partial a} \right)_{a=0} = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \stackrel{\text{def}}{=} U f$$

where

$$\begin{aligned} \left(\frac{\partial \phi}{\partial a} \right)_{a=0} &= \xi \\ \left(\frac{\partial \psi}{\partial a} \right)_{a=0} &= \eta \end{aligned}$$

Lie calls U the infinitesimal transformation. Further, because of the group composition law --which states that the product of two group trans-

formations is again a group transformation—

$$\left(\frac{\partial^2 f_1}{\partial a^2}\right)_{a=0} = U U f = U^2 f$$

and so on. Therefore we obtain the series

$$f_1 = f + U f a + U^2 f \frac{a^2}{2!} + \dots = e^{Ua} f \quad (1-3)$$

Hence in order for $f_1 = f$ for arbitrary x, y and a , that is, in order for f to be invariant, it is necessary and sufficient for $Uf = 0$ (since then $U^2 f = U U f = 0, U^3 f = U U U f = 0$, etc.).² It follows from this discussion that if

$$\xi = 0, \quad \eta^y = 0$$

for any particular point (x, y) then

$$x_1 = x, \quad y_1 = y$$

i.e., x and y are left invariant. In equation (1-3) U is said to generate the finite transformation; because of this U is called a generator in modern terminology.

Lie has shown that the relationship between x and y given by $f(x, y) = 0$ is covariant if and only if $Uf|_{f=0} = 0$.³

Now consider $f(x, y, \dot{y}, \dots, y^{(r)}) = 0$ where f is an infinitely differentiable function of $x, y, \dot{y}, \dots, y^{(r)}$. The transformation (1-1) acting on the arguments of f has the effect

$$f(x, y, \dot{y}, \dots, y^{(r)}) \rightarrow f(x_1, y_1, \dot{y}_1, \dots, y_1^{(r)}) \stackrel{df}{=} f_1(x, y, \dot{y}, \dots, y^{(r)}; a)$$

where $\dot{y}_1, \dots, y_1^{(r)}$ are the transformed derivatives. Assuming f_1 is analytic in a and expanding in a Taylor series around $a=0$ we obtain

$$f_1 = f + \left(\frac{\partial f_1}{\partial a}\right)_{a=0} a + \left(\frac{\partial^2 f_1}{\partial a^2}\right)_{a=0} \frac{a^2}{2!} + \dots$$

Now

$$\left(\frac{\partial f_1}{\partial a}\right)_{a=0} = \xi \frac{\partial f}{\partial x} + \eta^y \frac{\partial f}{\partial y} + \eta^{\dot{y}} \frac{\partial f}{\partial \dot{y}} + \dots + \eta^{y^{(r)}} \frac{\partial f}{\partial y^{(r)}} \stackrel{df}{=} U^{(r)} f$$

where

$$\eta^y \stackrel{df}{=} \frac{\partial}{\partial a} \left(\frac{dy}{dx} \right)_{a=0} = \frac{d\eta^y}{dx} - \dot{y} \frac{d\xi}{dx}$$

and in general

$$\eta^{y^{(q)}} = \frac{d\eta}{dx} y^{(q-1)} - y^{(q)} \frac{d\eta}{dx}$$

Again, since the group composition law still applies, we obtain the series

$$f_1 = f + U^{(r)} f a + U^{(r)^2} f \frac{a^2}{2!} + \dots = e^{U^{(r)} a} f$$

Therefore in order for $f_1 = f$ for $\{x, y, \dot{y}, \dots, y^{(r)}\}$ constrained by the equation $f=0$ and arbitrary a , it is necessary and sufficient for

$U^{(r)} f|_{f=0} = 0$ since adding variables to f does not alter the proof.⁵ It

follows that if for any particular case

$$\eta^{y^{(q)}} = 0$$

then $y_1^{(q)} = y^{(q)}$

$U^{(r)}$ is called the r^{th} extended infinitesimal transformation or generator in the case that the transformation is specified by (1-1).

Now consider the more general transformation

$$\begin{aligned} x_1 &= \lambda(x, y, \dot{y}, \dots, y^{(r-1)}; a) \\ y_1 &= \Theta(x, y, \dot{y}, \dots, y^{(r-1)}; a) \end{aligned} \quad (1-4)$$

where λ and Θ are continuous functions such that at $a=0$ the transformation reduces to the identity. We note that none of the equations given for the case of $f(x, y, \dot{y}, \dots, y^{(r)})$ is affected in form by the change from (1-1) to (1-4), as can be seen by letting $f(x, y) \rightarrow f(x, y, \dot{y}, \dots, y^{(r)})$.

The results presented in this work are based on a generalization of the Lie-Ovsjannikov theory⁶ presented by Anderson, Kumei and Wulfman.⁷ Fundamental to this generalization is the recognition that the sets of algebraically independent covariance transformation labels employed in the Lie-Ovsjannikov theory are not in general maximal sets.

The application of this generalization to an r^{th} order ordinary differential equation yields the result that the maximal set of algebraically in-

dependent labels for a transformation is $\{x, y, \dot{y}, \dots, y^{(r)}\}$. The possibility of using higher derivatives in the transformation laws for ordinary differential equations is well known for second order ordinary differential equations in the Hamiltonian Formulation of Classical Mechanics⁸ and it appears in Ovsjannikov's⁹ important generalization of Lie's theory. In the latter case it is introduced as a device to pass from an n^{th} order ordinary differential equation to an equivalent set of first order quasi-linear equations in order to avoid consideration of high extensions of Lie's theory. But, as has been previously pointed out¹⁰, the implications of using the maximal set fundamentally alters the theory of the hierarchy of extensions as originally proposed by Lie, both for systems of partial differential equations and of ordinary differential equations.

In this work I refer to the case involving the transformation law (1-1) as the Lie or Lie-Ovsjannikov case or method, and the case involving the transformation law (1-4) as the AKW extension, case or method.

Lie has shown¹¹ that for his case the commutators of the generators which leave any given differential equation covariant must close under a finite number of commutations, that is, each commutator must produce a linear combination of generators of the set. In the AKW case, however, it is shown by example that in general the set of generators which leave any differential equation covariant may not close under a finite number of commutations, i.e., the set may form an "infinite parameter Lie algebra." This is the terminology that has been recently adopted to describe a transformation group characterized by an infinite number of parameters and represents an extension of the original terminology of Lie to this

important physical case. Several examples of infinite parameter groups are given in this work. Thus the commutators of a set of AKW generators may produce new generators which must also leave the equation covariant.

Addressing ourselves now to the question of understanding the role and utility of the AKW extension, we are led to divide any set of generators (whether they close or not and/or involve derivative labels or not) into two mutually exclusive classes: those that generate non-trivial solutions directly by integration for an initial value problem and those that do not. The former further divide themselves into two further subclasses: those that generate the general solution of the original solution of the original differential equation and those that do not. Each of the latter can then also be used in principle to generate potentially new solutions through its action on solutions that it cannot generate directly by integration but which are known by other means, e.g. by integrating one of the other generators. Those that do not directly generate a solution still yield a distinctly new solution when operating on any known solution. What is interesting and extremely useful is that the generators which involve the additional derivative labels are also of both types.

The criterion that a generator directly yields by integration a solution is that the initial conditions for the original solution remain a set of initial conditions for the transformed solution, i.e., the original differential equation must not only be covariant under the action of the generator in question, but in addition the initial conditions must be invariant under its action.

Thus the equation

$$y_1(x_1) = y(x_1) \quad (1-5)$$

must apply if one is to remain on an integral curve, that is, if one is to generate solutions. Equation (1-5) implies

$$\left. \frac{\partial y_1(x_1)}{\partial a} \right|_{a=0} = \left. \frac{\partial y(x_1)}{\partial a} \right|_{a=0}$$

Therefore

$$\eta^y = \dot{y} \xi \quad (1-6)$$

This has been extensively discussed by Bluman and Cole¹² for partial differential equations in the Lie-Ovsjannikov theory. Note that in Lie's original treatment ξ and η^y were functions of only x and y and this was directly a quasi-linear first order ordinary differential equation, but in the AKW extension this is not the case because ξ and η^y in general involve \dot{y} .

Thus if the solution of (1-6) for a given generator satisfies the original differential equation, such a generator will yield a solution directly by integration.

The theorems given in this work apply only to ordinary differential equations which are integrable "in the large," i.e., such that one and only one integral curve of the differential equation passes through any two points in the (x, y) plane if the equation is of second order, and one and only one integral curve passes through any two points in the (x, y) plane if the values of $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ at one of the points are specified if the equation is of r^{th} order ($r \geq 2$). As emphasized by Gel'fand and Fomin, the conditions for the existence and uniqueness of such integral curves do not simply reduce to the usual existence theorems for differential equations, and they cite Bernstein's theorem¹³ for one set of sufficient conditions. But Bernstein's theorem is not sufficient to cover all known

cases of integrability "in the large," e.g. even the case $\ddot{y} = \text{constant}$, hence at this point I cannot further characterize integrability "in the large."¹⁴

In this work

$$\partial_x = \frac{\partial}{\partial x}$$

and an over-dot represents a derivative with respect to x --e.g. $\dot{y} = \frac{dy}{dx}$.

CHAPTER II: Mathematical Preliminaries

SECTION 1: Derivation of $U''(\dot{y}-f)|_{\dot{y}=f}=0$ for Second Order Ordinary Differential Equations

Consider

$$\ddot{y} = f(x, y, \dot{y}) \quad (1-1)$$

where f is arbitrary but its explicit \dot{y} dependence is known. In order to prove a result which will be used in the new theorems, we consider here a generator with a ξ and η^y of the form used in the theorems, i.e.,

$$\xi = \sum_{i=0}^m l_i(x, y) (\dot{y})^i, \quad \eta^y = \sum_{i=0}^m l_i(x, y) (\dot{y})^i \quad (1-2)$$

$$\text{Define: } T_i = l_{i-h}(i-1) \quad (i=1, 2, \dots, q) \quad (1-3)$$

where $q = n+1$ if $m \leq n$ and m if $m > n$, and

$$T_0 = l_0$$

If the index on T runs higher than the index on either l or h , such an l or h is null and T_i is only $-h(i-1)$ or l_i , respectively.

Equations (1-1) and (1-2) imply

$$\begin{aligned} \eta^{\dot{y}} &= \frac{d\eta^y}{dx} - \dot{y} \frac{d\xi}{dx} = \sum_{i=0}^m l_{ix} \dot{y}^i + \sum_{i=0}^m l_{iy} \dot{y}^{(i+1)} + \sum_{i=0}^m i l_i \dot{y}^{(i-1)} \dot{y} - \sum_{i=0}^m l_{ix} \dot{y}^{(i+1)} \\ &\quad - \sum_{i=0}^m l_{iy} \dot{y}^{(i+2)} - \sum_{i=0}^m i l_i \dot{y}^i \dot{y} = \sum_{i=0}^q T_{ix} \dot{y}^i + \sum_{i=0}^q T_{iy} \dot{y}^{(i+1)} + \left[\sum_{i=0}^m i l_i \dot{y}^{(i-1)} \right. \\ &\quad \left. - \sum_{i=0}^m i l_i \dot{y}^i \right] \dot{y} \end{aligned}$$

Therefore

$$\begin{aligned} \eta^{\ddot{y}} &= \frac{d\eta^{\dot{y}}}{dx} - \dot{y} \frac{d\eta^y}{dx} = \sum_{i=0}^q T_{ixx} \dot{y}^i + \sum_{i=0}^q 2 T_{ixy} \dot{y}^{(i+1)} + \sum_{i=0}^q T_{iyy} \dot{y}^{(i+2)} + \left[\sum_{i=0}^q i T_{ix} \dot{y}^{(i-1)} \right. \\ &\quad \left. + \sum_{i=0}^q (i+1) T_{iy} \dot{y}^i + \sum_{i=0}^m i l_{ix} \dot{y}^{(i-1)} - \sum_{i=0}^m (i+1) l_{ix} \dot{y}^i + \sum_{i=0}^m i l_{iy} \dot{y}^i + \sum_{i=0}^m (i+1) l_{iy} \dot{y}^{(i+1)} \right] \dot{y} \\ &\quad + \left[\sum_{i=0}^m (i^2 - i) l_i \dot{y}^{(i-2)} - \sum_{i=0}^m (i^2 + i) l_i \dot{y}^{(i-1)} \right] \dot{y}^2 + \left[\sum_{i=0}^m i l_i \dot{y}^{(i-1)} - \sum_{i=0}^m i l_i \dot{y}^i \right] \dot{y}^2 \end{aligned}$$

$$\begin{aligned}
 & + \left[\sum_{i=0}^{\infty} i l_i \dot{y}^i - \sum_{i=0}^{\infty} i h_i \dot{y}^{(i+1)} \right] f_y + \left[\sum_{i=0}^{\infty} i l_i \dot{y}^{(i-1)} - \sum_{i=0}^{\infty} i h_i \dot{y}^i \right] f f_y \\
 & = \sum_{i=0}^{\infty} T_{ixx} \dot{y}^i + \sum_{i=0}^{\infty} 2 T_{ixy} \dot{y}^{(i+1)} + \sum_{i=0}^{\infty} T_{iyy} \dot{y}^{(i+2)} + \left[\sum_{i=0}^{\infty} i T_{ix} \dot{y}^{(i-1)} + \sum_{i=0}^{\infty} (i+1) T_{iy} \dot{y}^i \right. \\
 & + \sum_{i=0}^{\infty} i T_{ix} \dot{y}^{(i-1)} + \sum_{i=0}^{\infty} i T_{iy} \dot{y}^i \left. \right] f + \left[\sum_{i=0}^{\infty} (i^2 - i) T_i \dot{y}^{(i-2)} \right] f^2 + \left[\sum_{i=0}^{\infty} i l_i \dot{y}^{(i-1)} \right. \\
 & - \sum_{i=0}^{\infty} i h_i \dot{y}^i \left. \right] f_x + \left[\sum_{i=0}^{\infty} i l_i \dot{y}^i - \sum_{i=0}^{\infty} i h_i \dot{y}^{(i+1)} \right] f_y + \left[\sum_{i=0}^{\infty} i l_i \dot{y}^{(i-1)} - \sum_{i=0}^{\infty} i h_i \dot{y}^i \right] f f_y
 \end{aligned}$$

Thus we have the result

$$\begin{aligned}
 U''(\ddot{y} - f) \Big|_{\ddot{y}=f} & = \eta^{\ddot{y}} + \left[- \sum_{i=0}^{\infty} T_{ix} \dot{y}^i - \sum_{i=0}^{\infty} T_{iy} \dot{y}^{(i+1)} \right] f_y + \left[- \sum_{i=0}^{\infty} i l_i \dot{y}^{(i-1)} + \sum_{i=0}^{\infty} i h_i \dot{y}^i \right] f f_y \\
 & + \left[- \sum_{i=0}^{\infty} l_i \dot{y}^i \right] f_y + \left[- \sum_{i=0}^{\infty} h_i \dot{y}^i \right] f_x = \sum_{i=0}^{\infty} T_{ixx} \dot{y}^i + \sum_{i=0}^{\infty} 2 T_{ixy} \dot{y}^{(i+1)} + \sum_{i=0}^{\infty} T_{iyy} \dot{y}^{(i+2)} \\
 & + \left[\sum_{i=0}^{\infty} 2 i T_{ix} \dot{y}^{(i-1)} + \sum_{i=0}^{\infty} (2i+1) T_{iy} \dot{y}^i \right] f + \left[\sum_{i=0}^{\infty} (i^2 - i) T_i \dot{y}^{(i-2)} \right] f^2 + \left[- \sum_{i=0}^{\infty} T_{ix} \dot{y}^i \right. \\
 & - \sum_{i=0}^{\infty} T_{iy} \dot{y}^{(i+1)} \left. \right] f_y + \left[\sum_{i=0}^{\infty} (i-1) T_i \dot{y}^i \right] f_y + \left[\sum_{i=0}^{\infty} i T_i \dot{y}^{(i-1)} \right] f_x = 0
 \end{aligned}$$

Knowing the \dot{y} dependence of f explicitly we can solve for l_0 , any l 's or h 's that are not paired and for the $l_i - h_{(i-1)}$ pairs where neither of the functions is zero. In other words it is only the difference $l_i - h_{(i-1)}$ which is uniquely determined by the equation given above and therefore precisely one function of the pair $l_i, h_{(i-1)}$ must be specified a priori or nondenumerably infinite classes of generators involving arbitrary functions would appear.

SECTION 2: Derivation of $U'''(\ddot{y}-f) \Big|_{\ddot{y}=f} = 0$ for Third Order Ordinary Differential Equations

Consider

$$\ddot{y} = f(x, y, \dot{y}, \ddot{y}) \quad (2-1)$$

where f is arbitrary but its \dot{y} and \ddot{y} dependence are known. Set

$$\begin{aligned}
 \xi & = \sum_{i,j=0}^{\infty} h_{ij}(x, y) \dot{y}^i \ddot{y}^j \\
 \eta & = \sum_{i,j=0}^{\infty} l_{ij}(x, y) \dot{y}^i \ddot{y}^j
 \end{aligned} \quad (2-2)$$

Define

$$T_{ij} = l_{ij} - h_{(i-1)j} \quad (i, j=1, 2, \dots, q)$$

where q is $n+1$ for i and n for j if $m \leq n$, and m for both i and j if $m > n$.

Also

$$T_{0j} = l_{0j}$$

If the indices on T run higher than the indices on either l or h , such an l or h is null and T_i is only $-h_{(i-1)j}$ or l_{ij} , respectively.

Equations (2-1) and (2-2) imply

$$\begin{aligned} \eta^j = \frac{d\eta^j}{dx} - \eta^j \frac{d\eta^j}{dx} &= \sum_{i=0}^m l_{ijxx} \eta^i \eta^j - \sum_{i=0}^m h_{ijxx} \eta^{(i+1)} \eta^j + \sum_{i=0}^m l_{ijxy} \eta^{(i+1)} \eta^j \\ &- \sum_{i=0}^m h_{ijxy} \eta^{(i+2)} \eta^j + \sum_{i=0}^m i l_{ijy} \eta^{(i-1)} \eta^{(i+1)} - \sum_{i=0}^m i h_{ijy} \eta^{(i)} \eta^{(i+1)} + \left[\sum_{i=0}^m j l_{ij} \eta^i \eta^{(j-1)} \right. \\ &\left. - \sum_{i=0}^m j h_{ij} \eta^{(i+1)} \eta^{(j-1)} \right] \eta^j \end{aligned}$$

Therefore

$$\begin{aligned} \eta^j = \frac{d\eta^j}{dx} - \eta^j \frac{d\eta^j}{dx} &= \sum_{i=0}^m l_{ijxx} \eta^i \eta^j - \sum_{i=0}^m h_{ijxx} \eta^{(i+1)} \eta^j + \sum_{i=0}^m 2 l_{ijxy} \eta^{(i+1)} \eta^j \\ &- \sum_{i=0}^m 2 h_{ijxy} \eta^{(i+2)} \eta^j + \sum_{i=0}^m l_{ijyy} \eta^{(i+2)} \eta^j - \sum_{i=0}^m h_{ijyy} \eta^{(i+3)} \eta^j + \sum_{i=0}^m 2 i l_{ijx} \eta^{(i-1)} \eta^{(i+1)} \\ &- \sum_{i=0}^m 2(i+1) h_{ijx} \eta^{(i)} \eta^{(i+1)} + \sum_{i=0}^m (2i+1) l_{ijy} \eta^{(i-1)} \eta^{(i+1)} - \sum_{i=0}^m (2i+3) h_{ijy} \eta^{(i)} \eta^{(i+1)} \\ &+ \sum_{i=0}^m i(i-1) l_{ij} \eta^{(i-2)} \eta^{(i+2)} - \sum_{i=0}^m (i^2+i) h_{ij} \eta^{(i-1)} \eta^{(i+2)} + \left[\sum_{i=0}^m 2j l_{ijx} \eta^i \eta^{(j-1)} \right. \\ &- \sum_{i=0}^m 2j h_{ijx} \eta^{(i+1)} \eta^{(j-1)} + \sum_{i=0}^m 2j l_{ijy} \eta^{(i+1)} \eta^{(j-1)} - \sum_{i=0}^m 2j h_{ijy} \eta^{(i+2)} \eta^{(j-1)} \\ &+ \sum_{i=0}^m i(2j+1) l_{ij} \eta^{(i-1)} \eta^{(j)} - \sum_{i=0}^m (2j+2i+1) h_{ij} \eta^i \eta^{(j)} \left. \right] \eta^j + \left[\sum_{i=0}^m j(j-1) l_{ij} \eta^i \eta^{(j-2)} \right. \\ &- \sum_{i=0}^m j(j-1) h_{ij} \eta^{(i+1)} \eta^{(j-2)} \left. \right] \eta^j + \left[\sum_{i=0}^m j l_{ij} \eta^i \eta^{(j-1)} - \sum_{i=0}^m j h_{ij} \eta^{(i+1)} \eta^{(j-1)} \right] \eta^j \\ &+ \left[\sum_{i=0}^m j l_{ij} \eta^{(i+1)} \eta^{(j-1)} - \sum_{i=0}^m j h_{ij} \eta^{(i+2)} \eta^{(j-1)} \right] \eta^j + \left[\sum_{i=0}^m j l_{ij} \eta^i \eta^{(j)} - \sum_{i=0}^m j h_{ij} \eta^{(i+1)} \eta^{(j)} \right] \eta^j \\ &+ \left[\sum_{i=0}^m j l_{ij} \eta^i \eta^{(j-1)} - \sum_{i=0}^m j h_{ij} \eta^{(i+1)} \eta^{(j-1)} \right] \eta^j \end{aligned}$$

Thus

$$\begin{aligned} \eta^j = \frac{d\eta^j}{dx} - \eta^j \frac{d\eta^j}{dx} &= \sum_{i=0}^m l_{ijxx} \eta^i \eta^j - \sum_{i=0}^m h_{ijxx} \eta^{(i+1)} \eta^j + \sum_{i=0}^m l_{ijxy} \eta^{(i+1)} \eta^j \\ &- \sum_{i=0}^m 2 h_{ijxy} \eta^{(i+2)} \eta^j + \sum_{i=0}^m 2 l_{ijyy} \eta^{(i+2)} \eta^j - \sum_{i=0}^m 2 h_{ijyy} \eta^{(i+3)} \eta^j + \sum_{i=0}^m l_{ijyy} \eta^{(i+3)} \eta^j \\ &- \sum_{i=0}^m h_{ijyy} \eta^{(i+4)} \eta^j + \sum_{i=0}^m 3 i l_{ijx} \eta^{(i-1)} \eta^{(i+1)} - \sum_{i=0}^m 3(i+1) h_{ijx} \eta^i \eta^{(i+1)} \\ &+ \sum_{i=0}^m (6i+3) l_{ijxy} \eta^i \eta^{(i+1)} - \sum_{i=0}^m (6i+4) h_{ijxy} \eta^{(i+1)} \eta^{(i+1)} + \sum_{i=0}^m 3(i+1) l_{ijyy} \eta^{(i+1)} \eta^{(i+1)} \end{aligned}$$

$$-\sum_{j=0}^m h_{ij} \dot{y}^{(i+1)} \ddot{y}^{(j-1)}] f \dot{f}^2 + \left[\sum_{j=0}^m l_{ij} \dot{y}^i \ddot{y}^{(j-1)} - \sum_{j=0}^m h_{ij} \dot{y}^{(i+1)} \ddot{y}^{(j-1)} \right] f^2 \dot{f} \ddot{y} \dot{y}$$

Hence we have

$$\begin{aligned} U'''(\ddot{y} - f(x, y, \dot{y}, \ddot{y})) &= \eta^{\ddot{y}} - \eta^f f_x - \eta^y f_y - \eta^{\dot{y}} \dot{f}_y - \eta^{\ddot{y}} \dot{f}_{\ddot{y}} \\ &= \sum_{j=0}^9 T_{ijxx} \dot{y}^i \ddot{y}^j + \sum_{j=0}^9 3 T_{ijxy} \dot{y}^{(i+1)} \ddot{y}^j + \sum_{j=0}^9 3 T_{ijyy} \dot{y}^{(i+2)} \ddot{y}^j + \sum_{j=0}^9 T_{ijyy} \dot{y}^{(i+3)} \ddot{y}^j \\ &+ \sum_{j=0}^9 3i T_{ijxx} \dot{y}^{(i-1)} \ddot{y}^{(j+1)} + \sum_{j=0}^9 (6i+3) T_{ijxy} \dot{y}^i \ddot{y}^{(j+1)} + \sum_{j=0}^9 3(i+1) T_{ijyy} \dot{y}^{(i+1)} \ddot{y}^{(j+1)} \\ &+ \sum_{j=0}^9 3i(i-1) T_{ijxx} \dot{y}^{(i-2)} \ddot{y}^{(j+2)} + \sum_{j=0}^9 3i^2 T_{ijxy} \dot{y}^{(i-1)} \ddot{y}^{(j+2)} + \sum_{j=0}^9 i(i-1)(i-2) T_{ijyy} \dot{y}^{(i-3)} \ddot{y}^{(j+3)} \\ &+ \sum_{j=0}^9 3j T_{ijxx} \dot{y}^i \ddot{y}^{(j-1)} + \sum_{j=0}^9 6j T_{ijxy} \dot{y}^{(i+1)} \ddot{y}^{(j-1)} + \sum_{j=0}^9 3j T_{ijyy} \dot{y}^{(i+2)} \ddot{y}^{(j-1)} \\ &+ \sum_{j=0}^9 3i(2j+1) T_{ijxx} \dot{y}^{(i-1)} \ddot{y}^j + \sum_{j=0}^9 (6ij+3i+3j+1) T_{ijxy} \dot{y}^i \ddot{y}^j + \sum_{j=0}^9 3i(i+j-1) T_{ijyy} \dot{y}^{(i-2)} \ddot{y}^{(j+1)}] f \\ &+ \sum_{j=0}^9 3j(j-1) T_{ijxx} \dot{y}^i \ddot{y}^{(j-2)} + \sum_{j=0}^9 3j(j-1) T_{ijxy} \dot{y}^{(i+1)} \ddot{y}^{(j-2)} + \sum_{j=0}^9 3i^2 T_{ijyy} \dot{y}^{(i-1)} \ddot{y}^{(j-1)}] f^2 \\ &+ \left[\sum_{j=0}^9 j(j-1)(j-2) T_{ij} \dot{y}^i \ddot{y}^{(j-3)} \right] f^3 + \left[\sum_{j=0}^9 3j T_{ijxx} \dot{y}^i \ddot{y}^{(j-1)} + \sum_{j=0}^9 3j T_{ijyy} \dot{y}^{(i+1)} \ddot{y}^{(j-1)} \right. \\ &+ \sum_{j=0}^9 (3ij+j+i-1) T_{ij} \dot{y}^i \ddot{y}^j] f_y + \left[\sum_{j=0}^9 (3j-1) T_{ijxx} \dot{y}^i \ddot{y}^j + \sum_{j=0}^9 (3j-1) T_{ijyy} \dot{y}^{(i+1)} \ddot{y}^j \right. \\ &+ \sum_{j=0}^9 3ij T_{ij} \dot{y}^{(i-1)} \ddot{y}^{(j+1)}] f_{yy} + \left[\sum_{j=0}^9 T_{ijxx} \dot{y}^i \ddot{y}^j - \sum_{j=0}^9 2 T_{ijxy} \dot{y}^{(i+1)} \ddot{y}^j - \sum_{j=0}^9 T_{ijyy} \dot{y}^{(i+2)} \ddot{y}^j \right. \\ &- \sum_{j=0}^9 2i T_{ijxx} \dot{y}^{(i-1)} \ddot{y}^{(j+1)} - \sum_{j=0}^9 (2i+1) T_{ijxy} \dot{y}^i \ddot{y}^{(j+1)} - \sum_{j=0}^9 i(i-1) T_{ijyy} \dot{y}^{(i-2)} \ddot{y}^{(j+2)}] f_{\ddot{y}} \\ &+ \left[\sum_{j=0}^9 j T_{ij} \dot{y}^i \ddot{y}^{(j-1)} \right] f_{xx} + \left[\sum_{j=0}^9 2j T_{ij} \dot{y}^{(i+1)} \ddot{y}^{(j-1)} \right] f_{xy} + \left[\sum_{j=0}^9 2j T_{ij} \dot{y}^i \ddot{y}^j \right] f_{x\ddot{y}} \\ &+ \left[\sum_{j=0}^9 j T_{ij} \dot{y}^{(i+2)} \ddot{y}^{(j-1)} \right] f_{yy} + \left[\sum_{j=0}^9 2j T_{ij} \dot{y}^{(i+1)} \ddot{y}^j \right] f_{y\ddot{y}} + \left[\sum_{j=0}^9 j T_{ij} \dot{y}^i \ddot{y}^{(j+1)} \right] f_{\ddot{y}\ddot{y}} \\ &+ \left[\sum_{j=0}^9 3j(j-1) T_{ij} \dot{y}^i \ddot{y}^{(j-2)} \right] f f_{xx} + \left[\sum_{j=0}^9 3j(j-1) T_{ij} \dot{y}^{(i+1)} \ddot{y}^{(j-2)} \right] f f_{xy} \\ &+ \left[\sum_{j=0}^9 3j(j-1) T_{ij} \dot{y}^i \ddot{y}^{(j-1)} \right] f f_y + \left[\sum_{j=0}^9 j T_{ijxx} \dot{y}^i \ddot{y}^{(j-1)} + \sum_{j=0}^9 j T_{ijyy} \dot{y}^{(i+1)} \ddot{y}^{(j-1)} \right. \\ &+ \sum_{j=0}^9 j T_{ij} \dot{y}^{(i-1)} \ddot{y}^j] f f_{\ddot{y}} + \left[\sum_{j=0}^9 2j T_{ij} \dot{y}^i \ddot{y}^{(j-1)} \right] f f_{xx} + \left[\sum_{j=0}^9 2j T_{ij} \dot{y}^{(i+1)} \ddot{y}^{(j-1)} \right] f f_{xy} \\ &+ \left[\sum_{j=0}^9 2j T_{ij} \dot{y}^i \ddot{y}^j \right] f f_{x\ddot{y}} + \left[\sum_{j=0}^9 2j(j-1) T_{ij} \dot{y}^i \ddot{y}^{(j-2)} \right] f^2 f_y + \left[\sum_{j=0}^9 j T_{ij} \dot{y}^i \ddot{y}^{(j-1)} \right] f^2 f_{\ddot{y}} = 0 \end{aligned}$$

Knowing the \dot{y} and \ddot{y} dependence of f explicitly, we can solve for l_0 , any l 's or h 's that are not paired and for the $l_{ij} - h_{(i-1)j}$ pairs where neither of the functions is zero. In other words it is only the difference $l_{ij} - h_{(i-1)j}$ which is uniquely determined by the equation given above and therefore precisely one function of the pair $l_{ij}, h_{(i-1)j}$ must be specified a priori or nondenumerably infinite classes of genera-

tors involving arbitrary functions would appear.

SECTION 3: Vandermonde Determinants¹⁵

Vandermonde determinants are used often in the theorems given in the following chapters. For this reason one of the special properties of a Vandermonde determinant ---that it does not vanish when all of its terms are different--- is explained here in some detail, although the proof is elementary.

A Vandermonde determinant of order n is a determinant each of whose rows consists of powers of some number from the $(n-1)^{st}$ power down to the power zero.

Theorem: A Vandermonde determinant of order n

$$D_n = \begin{vmatrix} y_1^{(n-1)} & y_1^{(n-2)} & \dots & y_1 & 1 \\ y_2^{(n-1)} & y_2^{(n-2)} & \dots & y_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_m^{(n-1)} & y_m^{(n-2)} & \dots & y_m & 1 \end{vmatrix}$$

can be written in the form

$$D_n = (y_1 - y_2)(y_1 - y_3) \dots (y_1 - y_m) \times \\ \times (y_2 - y_3) \dots \times \\ \times \dots \times \\ \times (y_{(n-1)} - y_m) \quad (3-1)$$

Proof: Assume the theorem holds true for D_k . Take D_{k+1} and substitute the first row for an unknown y .

$$D_{k+1}(y) = \begin{vmatrix} y^{(k)} & y^{(k-1)} & \dots & 1 \\ y_2^{(k)} & y_2^{(k-1)} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ y_{(k+1)}^{(k)} & y_{(k+1)}^{(k-1)} & \dots & 1 \end{vmatrix}$$

Expanding with respect to the elements of the first row, we see that

D_{k+1} is a polynomial of degree k in y . If we substitute $y=y_i$ ($i=2, \dots, k+1$) in the determinant, then the first row becomes the same as one of the other rows and the determinant vanishes. Thus the polynomial $D_{k+1}(y)$ has the roots y_i ($i=2, \dots, k+1$) and can be written in the form

$$D_{k+1}(y) = A_k \prod_{i=2}^{k+1} (y - y_i)$$

where A_k is the coefficient of y^k , i.e., the cofactor of the element

$y^{(k)}$ which stands in the upper left-hand corner of $D_{k+1}(y)$. But then A_k

is of the same form as D_k . Therefore A_k can be written as (3-1). Sub-

stituting (3-1) for A_k and y_1 for y we obtain

$$D_{k+1} = (y_1 - y_2)(y_1 - y_3) \dots (y_1 - y_k) \times \\ \times \dots \times \\ \times (y_k - y_{(k+1)})$$

Now

$$D_2 = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} = (y_1 - y_2)$$

Therefore, since the theorem is true for $n=2$ it is, by induction, true for all n .

Note that (3-1) implies that if all of the y_i 's ($i=1, 2, \dots, n$) are different, D_n cannot be zero.

CHAPTER III: Counting Theorems for Ordinary Differential
Equations of Second Order

SECTION 1: Lie's Counting Theorem for Ordinary Differential Equations
of Second Order

This theorem, which applies only to cases where ξ and η^y are functions of x and y only, was published by Sophus Lie in his book

Vorlesungen über Continuerliche Gruppen in 1893.¹⁶ Lie's theorem is given in some detail, as the generalization of it to include the AKW case is patterned closely after Lie's original. The proof is by contradiction.

Theorem: A differential equation of second order integrable "in the large" cannot be left covariant by more than eight linearly independent infinitesimal transformations of the form

$$U = \xi(x, y) \partial_x + \eta^y(x, y) \partial_y + \eta^{\dot{y}}(x, y, \dot{y}) \partial_{\dot{y}} + \eta^{\ddot{y}}(x, y, \dot{y}, \ddot{y}) \partial_{\ddot{y}}$$

Proof: Assume that the equation

$$\ddot{y} = F(x, y, \dot{y}) \tag{1-1}$$

is left covariant by nine linearly independent infinitesimal transformations U_1, U_2, \dots, U_9 . It is therefore also covariant under a linear combination of those transformations

$$U = a_1 U_1 + a_2 U_2 + \dots + a_9 U_9 = \xi \partial_x + \eta^y \partial_y + \eta^{\dot{y}} \partial_{\dot{y}} + \eta^{\ddot{y}} \partial_{\ddot{y}} \tag{1-2}$$

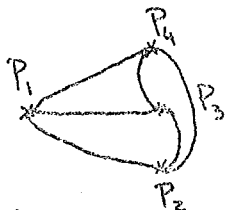
for all possible choices of the constants a_1, a_2, \dots, a_9 .

Select four points $P_i = (x_i, y_i)$, where $i=1,2,3,4$, such that each pair of points from this set determines a unique integral curve of (1-1) and no three of these points lie on the same integral curve. We now require that the four points P_1, P_2, P_3, P_4 remain invariant under the action of U , i.e., (see page 2) we set

$$\begin{aligned} a_1 \zeta_1(x_j, y_j) + \dots + a_9 \zeta_9(x_j, y_j) &= 0 \\ a_1 \eta_1^4(x_j, y_j) + \dots + a_9 \eta_9^4(x_j, y_j) &= 0 \end{aligned} \quad (j=1,2,3,4) \quad (1-3)$$

By means of these equations we can fix the ratios of eight of these constants to the ninth ---say, a_1, a_2, \dots, a_8 to a_9 . There must exist at least one set of four points such that the set of equations (1-3) yields finite non-trivial ratios for some eight of the a 's to the remaining one because of the assumed linear independence of the U_i 's.

An integral curve of a second order ordinary differential equation can be expressed as an equation with two arbitrary constants: fixing both constants fixes the curve. Therefore when we fix two points we have also fixed the integral curve between them. Thus when we set the points P_1, P_2, P_3, P_4 to be invariant under the action of U we automatically have that the six integral curves which join the different pairs of points are also invariant under the action of U . Hence through each of these points there pass at least three invariant integral curves.



This implies that $\zeta_j^4(x_j, y_j) = 0$ when evaluated at each of these points and every \dot{y} corresponding to an invariant integral curve through that point. In particular, introducing \dot{y}_{qj} for the slope of the integral curve passing

through points q and j evaluated at q ($q \neq j$), it is required for each point P_1, P_2, P_3, P_4 (represented by q) and all six curves (represented by j) passing through each pair of points that

$$\sum_{k=1}^4 a_k \eta_k^j = \sum_{k=1}^4 a_k \left[\frac{d\eta_k^j}{dx} - y \frac{d\eta_k^j}{dy} \right] = \sum_{k=1}^4 a_k \left[\frac{\partial \eta_k^j(q)}{\partial x} + \left(\frac{\partial \eta_k^j(q)}{\partial y} - \frac{\partial \eta_k^j(q)}{\partial x} y \right) y_{qj} - \frac{\partial \eta_k^j(q)}{\partial y} y_{qj}^2 \right] = 0$$

Letting b_{q1}, b_{q2}, b_{q3} be the values of $\frac{\partial \eta_1^j(q)}{\partial x}, \frac{\partial \eta_2^j(q)}{\partial y} - \frac{\partial \eta_2^j(q)}{\partial x} y, -\frac{\partial \eta_3^j(q)}{\partial y}$

respectively, we obtain the relations

$$\begin{aligned} b_{q3} y_{qj_1}^2 + b_{q2} y_{qj_1} + b_{q1} &= 0 \\ b_{q3} y_{qj_2}^2 + b_{q2} y_{qj_2} + b_{q1} &= 0 \\ b_{q3} y_{qj_3}^2 + b_{q2} y_{qj_3} + b_{q1} &= 0 \end{aligned}$$

The determinant of the coefficients

$$\begin{vmatrix} y_{qj_1}^2 & y_{qj_1} & 1 \\ y_{qj_2}^2 & y_{qj_2} & 1 \\ y_{qj_3}^2 & y_{qj_3} & 1 \end{vmatrix} = A$$

is a Vandermonde determinant (see page 13) and therefore, because three distinctly different integral curves were selected to pass through each point we have the result that $A \neq 0$, which implies that all the b 's are trivial, i.e.,

$$b_{q1} = b_{q2} = b_{q3} = 0 \quad (q=1,2,3,4)$$

which means that $\eta_k^j = 0$ for all integral curves through each of the four points P_1, P_2, P_3, P_4 . Therefore all these integral curves are invariant under the action of (1-2) where the a 's are subject to (1-3).

If P_5 is any fifth point in the region containing P_1, P_2, P_3, P_4 , it will lie on at least two integral curves each of which passes through one of those points. If P_5 does not lie upon any of the six integral curves determined by the four points it lies on four integral curves each of which passes through one of those points; if P_5 lies on

one of the integral curves determined by the four points, the number is three, and two if P_5 is at the intersection of two of these special integral curves. Since any integral curve which passes through one of the four points P_1, P_2, P_3, P_4 is invariant, P_5 lies on the intersection of at least two invariant curves and is therefore itself invariant. In this way, since P is arbitrary, every point in the region and therefore every point in the plane is found to be left invariant under the action of (1-2). The latter must therefore be identically zero, i.e.

$$a_1 U_1 + a_2 U_2 + \dots + a_9 U_9 = 0$$

for non-trivial a 's. Hence any nine infinitesimal transformations which leave an ordinary differential equation of second order covariant cannot be linearly independent.

Thus it has been shown that the number of linearly independent infinitesimal transformations which leave an ordinary differential equation of second order covariant must be less than nine. Lie uses an example ($\ddot{y}=0$) to show that at least one equation exists which has eight linearly independent infinitesimal transformations which leave it covariant.¹⁷ Hence it has been shown that the maximum number of linearly independent infinitesimal transformations (in which ξ and η are functions of x and y only) which leave an ordinary differential equation of second order covariant is eight.

SECTION 2: Overdamped Harmonic Oscillator (Lie case)*

* This example was worked independently as an exercise by several members (including the author) of a course in the Group-Theoretic Properties of Differential Equations conducted by Robert L. Anderson and Carl E. Wulfman in the Spring of 1972 at the University of the Pacific. These results have never before been published.

The overdamped harmonic oscillator produces the maximum number of generators allowed in the Lie case for second order ordinary differential equations. In this section those generators are derived.

$$\ddot{y} = -b\dot{y} - ky$$

$$U''(\dot{y} + b\dot{y} + ky) \Big|_{\dot{y} = -b\dot{y} - ky} = 0$$

According to Cohen¹⁸

$$U''(\ddot{y} - f(x, y, \dot{y})) = [(\eta_y^y - 2\xi_{xx})\dot{y} - \xi_x\dot{y} - \eta_y^y\dot{y} - \eta_x^y\dot{y} + \eta_{xx}^y] + [2\eta_{xy}^y - \xi_{xx} - 3\xi_y\dot{y} - (\eta_y^y - \xi_{xx})\dot{y}] \dot{y} + [\eta_{yy}^y - 2\xi_{xy} + \xi_y\dot{y}] \dot{y}^2 + [-\xi_{yy}] \dot{y}^3 = 0$$

Substituting $\ddot{y} = f = -b\dot{y} - ky$ we have

$$(\eta_y^y - 2\xi_{xx})(-b\dot{y} - ky) - \eta_y^y(-k) - \eta_x^y(-b) + \eta_{xx}^y[2\eta_{xy}^y - \xi_{xx} - 3\xi_y(-b\dot{y} - ky) - (\eta_y^y - \xi_{xx})(-b)] \dot{y} + [\eta_{yy}^y - 2\xi_{xy} + \xi_y(-b)] \dot{y}^2 - \xi_{yy} \dot{y}^3 = [-\xi_{yy}] \dot{y}^3 + [\eta_{yy}^y - 2\xi_{xy} + 2\xi_y b] \dot{y}^2 + [2\eta_{xy}^y - \xi_{xx} - 3\xi_y ky + \eta_y^y b - \xi_x b - \eta_y^y b + 2\xi_x b] \dot{y} + [-\eta_y^y ky + 2\xi_{xx} ky + \eta_x^y k + \eta_{xx}^y b + \eta_{xx}^y] = 0$$

Since \dot{y} is arbitrary, all the coefficients of the powers of \dot{y} must independently equal zero, i.e.,

$$-\xi_{yy} = 0 \quad (2-1)$$

$$\eta_{yy}^y - 2\xi_{xy} + 2\xi_y b = 0 \quad (2-2)$$

$$2\eta_{xy}^y - \xi_{xx} + 3\xi_y ky + \eta_y^y b - \xi_x b - \eta_y^y b + 2\xi_x b = 0 \quad (2-3)$$

$$-\eta_y^y ky + 2\xi_{xx} ky + \eta_x^y k + \eta_{xx}^y b + \eta_{xx}^y = 0 \quad (2-4)$$

The above are the determining equations.

$$(2-1) \Rightarrow \xi_y = a(x)y + c(x)$$

$$(2-2) \Rightarrow \eta_y^y = [a_x - a.b]y^2 + d(x)y + g(x)$$

$$(2-3) \Rightarrow [3a_{xx} - 3ba_x + 3ka_x]y + [2d_x - c_{xx} + c_x b] = 0$$

Therefore, since y is arbitrary,

$$a = \frac{P_1 x}{2} + \frac{P_2 x}{2} \quad \text{where} \quad P_{\pm} = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4k}}{2} \quad (2-5)$$

$$d_x = \frac{c_{xx}}{2} - \frac{c_x b}{2} \quad (2-6)$$

$$(2-4) \Rightarrow [kba + ka_x - b^2 a_x + a_{xxx}]y^2 + [2kc_x + bd_x + d_{xx}]y + [kg + bg_x + g_{xx}] = 0$$

Therefore, since y is arbitrary,

$$g = B_1 e^{m_+ x} + B_2 e^{m_- x} \text{ where } m_{\pm} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4k}}{2}$$

$$(2-4), (2-6) \Rightarrow c = A_1 e^{\gamma x} + A_2 e^{-\gamma x} + \lambda \text{ where } \gamma = \sqrt{b^2 - 4k} \quad (2-7)$$

$$(2-4), (2-6), (2-7) \Rightarrow d = A_1 \left[\frac{-b + \sqrt{b^2 - 4k}}{2} \right] e^{x\sqrt{b^2 - 4k}} + A_2 \left[\frac{-b - \sqrt{b^2 - 4k}}{2} \right] e^{-x\sqrt{b^2 - 4k}} + \beta$$

Hence

$$U = C_1 e^{P_+ x} \left[y \partial_x + P_+ y^2 \partial_y - b y^2 \partial_y \right] + C_2 e^{P_- x} \left[y \partial_x + P_- y^2 \partial_y - b y^2 \partial_y \right]$$

$$+ A_1 e^{x\sqrt{b^2 - 4k}} \left[\partial_x + \left(-\frac{b}{2} + \frac{\sqrt{b^2 - 4k}}{2} \right) y \partial_y \right] + A_2 e^{-x\sqrt{b^2 - 4k}} \left[\partial_x + \left(-\frac{b}{2} - \frac{\sqrt{b^2 - 4k}}{2} \right) y \partial_y \right]$$

$$+ B_1 e^{m_+ x} [\partial_y] + B_2 e^{m_- x} [\partial_y] + \lambda [\partial_x] + \beta [y \partial_y]$$

The eight extended operators are given in Table A. In the table $\delta = \frac{\sqrt{b^2 - 4k}}{2}$.

TABLE A

$U_1'' =$	$e^{\left(\frac{b}{2} + \delta\right)x} \left\{ y \partial_x + \left(-\frac{b}{2} + \delta\right) y^2 \partial_y + [-ky^2 + \left(-\frac{3}{2}b + \delta\right) y \partial_y - y^2] \partial_y + \left[\left(\frac{3}{2}b - \delta\right) ky^2 + \left(\frac{3}{2}b^2 + b\delta\right) y \partial_y + by^2 \right] \partial_y \right\}$
$U_2'' =$	$e^{\left(\frac{b}{2} - \delta\right)x} \left\{ y \partial_x + \left(-\frac{b}{2} - \delta\right) y^2 \partial_y + [-ky^2 + \left(-\frac{3}{2}b - \delta\right) y \partial_y - y^2] \partial_y + \left[\left(\frac{3}{2}b + \delta\right) ky^2 + \left(\frac{3}{2}b^2 + b\delta\right) y \partial_y + by^2 \right] \partial_y \right\}$
$U_3'' =$	$e^{2\delta x} \left\{ \partial_x + \left(-\frac{b}{2} + \delta\right) y \partial_y + \left[(-b\delta + \frac{b^2}{2} - 2k)y + \left(-\frac{b}{2} - \delta\right) y\right] \partial_y + \left[\left(-\frac{b^3}{2} + \frac{5}{2}bk + b^2\delta - k\delta\right)y + \left(b\delta + \frac{b^2}{2}\right) y\right] \partial_y \right\}$
$U_4'' =$	$e^{-2\delta x} \left\{ \partial_x + \left(-\frac{b}{2} - \delta\right) y \partial_y + \left[(b\delta + \frac{b^2}{2} - 2k)y + \left(-\frac{b}{2} + \delta\right) y\right] \partial_y + \left[\left(-\frac{b^3}{2} + \frac{5}{2}bk - b^2\delta + k\delta\right)y + \left(-b\delta + \frac{b^2}{2}\right) y\right] \partial_y \right\}$
$U_5'' =$	$e^{\left(-\frac{b}{2} + \delta\right)x} \left\{ \partial_y + \left(-\frac{b}{2} + \delta\right) \partial_y + \left(\frac{b^2}{2} - b\delta - k\right) \partial_y \right\}$
$U_6'' =$	$e^{\left(-\frac{b}{2} - \delta\right)x} \left\{ \partial_y + \left(-\frac{b}{2} - \delta\right) \partial_y + \left(\frac{b^2}{2} + b\delta - k\right) \partial_y \right\}$
$U_7'' =$	∂_x
$U_8'' =$	$y \partial_y + y \partial_y + y \partial_y$

The commutators of the generators listed in Table A are given in Table B. The generators enumerated in the column furthest left correspond to the first member of the commutator, while the generators enumerated in the top row correspond to the second. Thus we read the commutator $[U_1, U_4]$ from the third box from the left on the top row of boxes, obtaining the value $-2\delta U_2$. All other commutation tables given in this thesis are to be read in the same manner. Since $[A, B] = -[B, A]$ and $[A, A] = 0$, the values given in Table B are sufficient to define all possible commutators of the generators listed in Table A.

As in the previous table, $\delta = \frac{\sqrt{b^2 - 4k}}{2}$.

TABLE B

$[A, B]$	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
U_1		0	0	$-2\delta U_2$	U_3	$-U_7 - 3\delta U_8$	$(-\frac{b}{2} - \delta) U_1$	$-U_1$
U_2			$2\delta U_1$	0	$-U_7 + (\frac{b}{2} + \delta) U_8$	$-U_4$	$(-\frac{b}{2} + \delta) U_2$	$-U_2$
U_3				$-4\delta U_7 + 2b\delta U_8$	0	$-2\delta U_5$	$-2\delta U_3$	0
U_4					$(b + 2\delta) U_6$	0	$2\delta U_4$	0
U_5						0	$(\frac{b}{2} - \delta) U_5$	U_5
U_6							$(\frac{b}{2} + \delta) U_6$	U_6
U_7								0
U_8								

SECTION 3: Generalized Counting Theorem for Second Order Ordinary
Differential Equations¹⁹

The theorem given in this section is a generalization of Lie's counting theorem for second order ordinary differential equations. It applies to certain generators obtained by the AKW method and includes Lie's generators as a subcase.

Lie took ξ and η^y to depend solely on x and y . He did not need to assume any particular form for ξ or η^y . As is shown below, when one uses the AKW extension the y dependence of ξ and η^y must be chosen, although the x and y dependence is still arbitrary.

Consider: $\ddot{y} - f(x, y, \dot{y}) = 0$

$$\xi = \xi(x, y, \dot{y})$$

$$\eta^y = \eta^y(x, y, \dot{y})$$

Therefore

$$\eta^y \Big|_{\dot{y}=f} = \frac{d\eta^y}{dx} - \dot{y} \frac{d\xi}{dx} = (\eta^y_x) + (\eta^y_y - \xi_x) \dot{y} + (-\xi_y) \dot{y}^2 + [(\eta^y_{\dot{y}}) + (-\xi_{\dot{y}}) \dot{y}] f$$

$$\eta^{\dot{y}} \Big|_{\dot{y}=f} = \frac{d\eta^{\dot{y}}}{dx} - \dot{y} \frac{d\xi}{dx} = (\eta^{\dot{y}}_{xx}) + (2\eta^{\dot{y}}_{xy} - \xi_{xx}) \dot{y} + (-\xi_{xy}) \dot{y}^2 + (-\xi_{yy}) \dot{y}^3$$

$$+ [(2\eta^{\dot{y}}_{x\dot{y}} + \eta^{\dot{y}}_{\dot{y}\dot{y}} - \xi_{x\dot{y}}) + (2\eta^{\dot{y}}_{y\dot{y}} - 2\xi_{xy\dot{y}} - 2\xi_{y\dot{y}}) \dot{y} + (-2\xi_{y\dot{y}\dot{y}}) \dot{y}^2] f + [(\eta^{\dot{y}}_{\dot{y}\dot{y}} - \xi_{\dot{y}\dot{y}}) + (-\xi_{\dot{y}\dot{y}\dot{y}}) \dot{y}] f^2 + [(\eta^{\dot{y}}_{\dot{y}}) + (-\xi_{\dot{y}}) \dot{y}] f_x + [(\eta^{\dot{y}}_{\dot{y}}) \dot{y} + (-\xi_{\dot{y}}) \dot{y}^2] f_y + [(\eta^{\dot{y}}_{\dot{y}}) + (-\xi_{\dot{y}}) \dot{y}] f f_y$$

Thus

$$U''(\ddot{y} - f(x, y, \dot{y})) \Big|_{\dot{y}=f} = (\eta^y_{\dot{y}\dot{y}} - 2\xi_{x\dot{y}}) f - \xi f_{xx} - \eta^y f_{xy} - \eta^y_x f_{\dot{y}} + \eta^y_{xx}$$

$$+ [2\eta^y_{xy} - \xi_{xx} - 3\xi_{xy} f - (\eta^y_{\dot{y}} - \xi_x) f_{\dot{y}}] \dot{y} + (\eta^y_{yy} - 2\xi_{xy} + \xi_y f_{\dot{y}}) \dot{y}^2$$

$$- \xi_{yy} \dot{y}^3 + 2\eta^y_{x\dot{y}} f + \eta^y_{\dot{y}\dot{y}} f^2 + \eta^y_{\dot{y}} f_x - 2\xi_{\dot{y}} f^2 + [2\eta^y_{y\dot{y}} f + \eta^y_{\dot{y}} f_y$$

$$-2\xi_{xy}f - \xi_{yy}f^2 - \xi_y f_x \dot{y} + [-2\xi_{yy}f - \xi_y f_y] \dot{y}^2 = 0^* \quad (3-1)$$

Equation (3-1) is a partial differential equation in two arbitrary functions (assuming f is given for any specific case). For any particular η^y we obtain an infinity of possible ξ^x and vice versa, since specifying either one leaves a partial differential equation in one arbitrary function. Thus Ovsjannikov's result²⁰ that there exists an infinite number of one-parameter covariance transformations for ordinary differential equations of any order is again confirmed. Because of this infinity, counting theorems analogous to the one given in Section 2 can only be obtained if an explicit \dot{y} dependence is assumed for ξ and η^y . Here we derive such a new counting theorem by expanding the \dot{y} dependence of the various generators in terms of a subset of a particular linearly independent set of functions, namely simple powers of \dot{y} .

Theorem: A second order ordinary differential equation which is integrable "in the large" and is of the form

$$\ddot{y} = f(x, y, \dot{y}) \quad (3-2)$$

is covariant under the action of at most r linearly independent generators of the form

$$U_k = \xi_k \partial_x + \eta_k^y \partial_y + \eta_k^{\dot{y}} \partial_{\dot{y}} + \eta_k^{\ddot{y}} \partial_{\ddot{y}} \quad (3-3)$$

where, for fixed m and n , ξ and η^y are of the form

$$\xi_k = \sum_{i=0}^m h_{ik}(x, y) (\dot{y})^i \quad (3-4)$$

$$\eta_k^y = \sum_{i=0}^n l_{ik}(x, y) (\dot{y})^i$$

subject to the condition that in each pair $\xi_{ik}, h_{(i-1)k}$ ($i=1, \dots, q=\max\{m, n+1\}$, and all k) precisely one function is specified a priori

* The last equation is taken from the notes by Robert L. Anderson for the course previously cited.

(see page 3) and r is given by

$$r = \begin{cases} (m+n+2)(n+4) & \text{for } n \geq m, n \geq 0, m \geq 0 \\ (m+n+2)(m+3) & \text{for } n < m, n \geq 0, m \geq 0 \end{cases} \quad (3-5a)$$

$$(3-5b)$$

Proof: The proof is by contradiction. Fix n, m and assume that equation (3-2) remains covariant under the action of $r+1$ linearly independent generators U_1, U_2, \dots, U_{r+1} , where r is determined by (3-5). Therefore the equation is also covariant under the action of an arbitrary non-trivial linear combination of these generators

$$U = \sum_{k=1}^{r+1} a_k U_k \quad (3-6)$$

Select $s=r/(m+n+2)$ points $P_i = (x_i, y_i)$, where $i=1, 2, \dots, s$, such that each pair of points from this set determines a unique integral curve of (3-2) and no three of these points lie on the same integral curve. We now require that the s points P_1, P_2, \dots, P_s remain invariant under the action of U , i.e., (see page 2) we set

$$\begin{aligned} a_1 \xi_1(x_j, y_j) + a_2 \xi_2(x_j, y_j) + \dots + a_{r+1} \xi_{(r+1)}(x_j, y_j) &= 0 \\ a_1 \eta_1(x_j, y_j) + a_2 \eta_2(x_j, y_j) + \dots + a_{r+1} \eta_{(r+1)}(x_j, y_j) &= 0 \end{aligned}$$

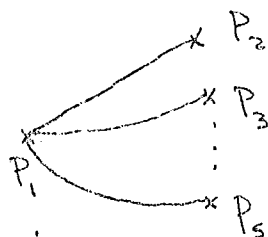
where $j=1, 2, \dots, s$. Since these equations must be true for any curve passing through any of the s fixed points, the set of equations given above must be true for arbitrary j , which implies the following system of r equations

$$\begin{aligned} a_1 h_{0,1}(x_j, y_j) + a_2 h_{0,2}(x_j, y_j) + \dots + a_{r+1} h_{0,r+1}(x_j, y_j) &= 0 \\ \vdots \\ a_1 h_{m,1}(x_j, y_j) + a_2 h_{m,2}(x_j, y_j) + \dots + a_{r+1} h_{m,r+1}(x_j, y_j) &= 0 \end{aligned} \quad (3-7)$$

$$\begin{aligned} a_1 l_{0,1}(x_j, y_j) + a_2 l_{0,2}(x_j, y_j) + \dots + a_{r+1} l_{0,r+1}(x_j, y_j) &= 0 \\ \vdots \\ a_1 l_{m,1}(x_j, y_j) + a_2 l_{m,2}(x_j, y_j) + \dots + a_{r+1} l_{m,r+1}(x_j, y_j) &= 0 \end{aligned} \quad (3-8)$$

where $j=1,2,\dots,s$. By means of these equations we can fix the ratios of r of these constants to the $(r+1)^{st}$ --say a_1, a_2, \dots, a_r to a_{r+1} .

An integral curve of a second order ordinary differential equation can be expressed as an equation with two arbitrary constants: fixing both constants fixes the curve. Therefore when two points remain invariant under the action of U (as constrained by (3-7) and (3-8)), the integral curve passing through them also remains invariant under its action. Thus when we set the s points P_i to be invariant we automatically have that the integral curves which join the different pairs of points P_i are also invariant under the action of U as constrained by (3-7) and (3-8). Hence through each point there pass at least $(s-1)$ invariant integral curves.



This implies that $\eta^j = 0$ when evaluated at each of these points and every \dot{y} corresponding to an invariant integral curve through that point. In particular, using \dot{y}_{qj} for the slope of the integral curve passing through the points q and j evaluated at q ($q+j$), and substituting (3-4) into the expression for η^j given on page 8, it is required for each point P_1, P_2, \dots, P_s (represented by q) and all $(s-1)$ curves (represented by j) passing through each point that

$$\begin{aligned} \sum_{k=1}^{r+1} a_k \eta_k^j(qj) &= \sum_{k=1}^{r+1} a_k \left[\frac{d\eta_k^j(qj)}{dx} - \dot{y}_{qj} \frac{d\eta_k^j(qj)}{dy} \right] = \sum_{k=1}^{r+1} a_k \left\{ \frac{d}{dx} \left[\sum_{i=0}^m l_{ik}(q) (\dot{y}_{qj})^i \right] \right. \\ &- \dot{y}_{qj} \frac{d}{dy} \left[\sum_{i=0}^m l_{ik}(q) (\dot{y}_{qj})^i \right] \Big\} = \sum_{k=1}^{r+1} a_k \left[\sum_{i=0}^m \left\{ \frac{\partial l_{ik}(q)}{\partial x} (\dot{y}_{qj})^i + \frac{\partial l_{ik}(q)}{\partial y} (\dot{y}_{qj})^{(i+1)} \right. \right. \\ &+ i l_{ik}(q) f(x, y, \dot{y}) (\dot{y}_{qj})^{i-1} \Big\} - \sum_{i=0}^m \left\{ \frac{\partial l_{ik}(q)}{\partial x} (\dot{y}_{qj})^{(i+1)} + \frac{\partial l_{ik}(q)}{\partial y} (\dot{y}_{qj})^{(i+2)} \right. \\ &\left. \left. + i l_{ik}(q) f(x, y, \dot{y}) (\dot{y}_{qj})^{i-1} \right\} \right] = 0 \end{aligned} \quad (3-9)$$

Equation (3-9) implies that the highest power of \dot{y} (s-2) is given by

I) $n+2$ (for $n \geq m$, $n \geq 0$, $m \geq 0$) II) $m+1$ (for $n < m$, $n \geq 0$, $m > 0$)

Note that equations (3-7) and (3-8) imply that the third and sixth terms of (3-9) are zero, thus eliminating any dependence on the form of (3-2).

Let $b_{q1}, b_{q2}, \dots, b_{q(s-1)}$ represent the coefficients of $(\dot{y}_{qj})^0, (\dot{y}_{qj})^1, \dots, (\dot{y}_{qj})^{(s-2)}$, respectively. Then (3-9) gives for each q and the $(s-1)$ j 's

$$\begin{aligned} b_{q(s-1)} (\dot{y}_{qj})^{(s-2)} + \dots + b_{q2} \dot{y}_{qj} + b_{q1} &= 0 \\ b_{q(s-1)} (\dot{y}_{qj(s-1)})^{(s-2)} + \dots + b_{q2} \dot{y}_{qj(s-1)} + b_{q1} &= 0 \end{aligned} \quad (3-10)$$

The determinant of the coefficients

$$\begin{vmatrix} \dot{y}_{qj_1}^{(s-2)} & \dots & \dot{y}_{qj_1} & 1 \\ \vdots & & \vdots & \vdots \\ \dot{y}_{qj(s-1)}^{(s-2)} & \dots & \dot{y}_{qj(s-1)} & 1 \end{vmatrix} = A \quad (3-11)$$

is a Vandermonde determinant (see page 13). Since $(s-1)$ distinctly different integral curves were selected to pass through each point, the \dot{y}_{qj} 's must all be different and therefore $A \neq 0$, which implies that all the b 's are trivial, i.e.,

$$b_{q1} = b_{q2} = \dots = b_{q(s-1)} = 0 \quad (q=1, 2, \dots, s)$$

which means that $\dot{y} = 0$ for all integral curves through each of the s points P_1, P_2, \dots, P_s . Therefore all these integral curves are invariant under the action of (3-6) where the a 's are subject to (3-7) and (3-8).

Because the minimum s possible is four, corresponding to Lie's original case $m=n=0$, if P_{s+1} is any $(s+1)^{st}$ point, it will lie on at least two distinct integral curves each of which passes through at least one of the s points. Since these curves have been shown to be invariant, P_{s+1} lies on the intersection of two invariant integral curves and must

be therefore itself invariant. Now P_{s+1} is arbitrary, so we have the result that all the points in the plane are left invariant under the action of (3-6) where the a 's are constrained by (3-7) and (3-8). Therefore (3-6) must be identically zero, i.e., the set U_1, U_2, \dots, U_{r+1} is linearly dependent, which contradicts the original assumption.

This argument does not apply to any linear combination of r or less generators of the assumed form. This is because the highest power $(s-2)$ of \dot{y} which depends on m, n and the properties of the Vandermonde determinant uniquely fix the number s of invariant points for which the argument is valid. Once this number is fixed, the number of generators which can be ostensibly assumed to be linearly independent must be such that not all the constants in their linear combination are determined by requiring the invariance of the points.

It has thus been shown that the maximum number of linearly independent generators of the form given in equations (3-3) and (3-4) is less than $r+1$. The fact that the maximum (r) can be achieved is shown by two examples given in Sections 4 and 5.

SECTION 4: Free Particle

$$\ddot{y}=0 \quad (4-1)$$

$$\begin{aligned} \eta^0 &= h_0(x, y) + h_1(x, y) \dot{y} \\ \eta^1 &= l_0(x, y) \end{aligned} \quad (4-2)$$

From the equation given on page 9 we obtain for $f=0, m=0, n=1$

$$\begin{aligned} U''(\ddot{y}) \Big|_{\ddot{y}=0} &= \eta^{\ddot{y}} = l_{0xx} - h_{0xx} \dot{y} - h_{1xx} \dot{y}^2 + 2l_{0xy} \dot{y} - 2h_{0xy} \dot{y}^2 - 2h_{1xy} \dot{y}^3 \\ &+ l_{0yy} \dot{y}^2 - h_{0yy} \dot{y}^3 - h_{1yy} \dot{y}^4 = (l_{0xx}) + (2l_{0xy} - h_{0xx}) \dot{y} + (l_{0yy} - 2h_{0xy} \dot{y} \\ &- h_{1yy} \dot{y}^4) \end{aligned}$$

$$-h_{1xx})\dot{y}^2 + (-h_{0yy} - 2h_{1xy})\dot{y}^3 + (-h_{1yy})\dot{y}^4 = 0$$

Therefore, since \dot{y} is arbitrary, we obtain the following system of determining equations

$$h_{0xx} = 0 \quad (4-3)$$

$$2h_{0xy} - h_{0xx} = 0 \quad (4-4)$$

$$h_{0yy} - 2h_{0xy} - h_{1xx} = 0 \quad (4-5)$$

$$-h_{0yy} - 2h_{1xy} = 0 \quad (4-6)$$

$$-h_{1yy} = 0 \quad (4-7)$$

$$(4-3) \Rightarrow h_0 = \alpha(y)x + \beta(y)$$

$$(4-7) \Rightarrow h_1 = \gamma(x)y + \delta(x)$$

$$(4-6) \Rightarrow h_0 = -\gamma_x y^2 + \lambda(x)y + \chi(x)$$

$$(4-4) \Rightarrow -\gamma_{xxx} y^2 + \lambda_{xx} y + \chi_{xx} = 2\alpha_y$$

Therefore, since α is a function of y only

$$\gamma_{xxx} = \text{const.} \Rightarrow \gamma = Ax^3 + Dx^2 + Ex + F$$

$$\lambda_{xx} = \text{const.} \Rightarrow \lambda = Bx^2 + Gx + H$$

$$\chi_{xx} = \text{const.} \Rightarrow \chi = Cx^2 + Ix + J$$

Therefore

$$2\alpha_y = (-6A)y^2 + (2B)y + (2C) \Rightarrow \alpha = -Ay^3 + \frac{B}{2}y^2 + Cy + K$$

$$(4-5) \Rightarrow 12Ax + \beta_{yy} + 6Dy - 3Bx - 2G - \delta_{xx} = 0$$

$$\text{Therefore } \beta_{yy} = (-12Ax - 6D)y + (3Bx + 2G + \delta_{xx})$$

Hence, since β is a function of y only

$$(-12A)x + (-6D) = \text{const.} \Rightarrow A = 0, \quad (3B)x + 2G + \delta_{xx} = \text{const.}$$

Thus

$$\delta_{xx} = (-3B)x + (-2G + 2L) \Rightarrow \delta = \left(-\frac{B}{2}\right)x^3 + (-G + L)x^2 + (Mx + N)$$

Therefore

$$\beta = -Dy^3 + Ly^2 + Py + Q$$

Hence

$$\begin{aligned} U = & B(x^2y \partial_x - \frac{x^3y}{2} \partial_x + \frac{xy^2}{2} \partial_y) + C(x^2 \partial_x + xy \partial_y) \\ & + D(-2xy^2 \partial_x + x^2y \partial_y - y^3 \partial_y) + E(-y^2 \partial_x + xy \partial_x) + F(y \partial_x) \\ & + G(xy \partial_x - x^2y \partial_x) + H(y \partial_x) + I(x \partial_x) + J(\partial_x) + K(x \partial_y) \\ & + L(x^2y \partial_x + y^2 \partial_y) + M(xy \partial_x) + N(y \partial_x) + P(y \partial_y) + Q(\partial_y) \end{aligned}$$

The extended operators for $\dot{y}=0$ are listed in Table C. Note that we recover Lie's eight generators²¹ if we use $U_9'' + U_8''$ and $U_9'' - U_8''$ instead of U_8'' and U_9'' . The first seven operators and $U_9'' + U_8''$ are the Lie operators; the rest (including $U_9'' - U_8''$) are new.

TABLE C

U_1''	$y \partial_x - y^2 \partial_y$
U_2''	$x \partial_x - y \partial_y$
U_3''	$x \partial_y + \partial_y$
U_4''	$y \partial_y + y \partial_y$
U_5''	∂_x
U_6''	∂_y
U_7''	$x^2 \partial_x + xy \partial_y + (y - xy) \partial_y$
U_8''	$(xy - x^2y) \partial_x + (-y \partial_y + xy^2) \partial_y$
U_9''	$x^2y \partial_x + y^2 \partial_y + (2y \partial_y - 2xy^2) \partial_y$
U_{10}''	$y \partial_x$
U_{11}''	$y \partial_x - y^3 \partial_y$

Table D is the commutation table of the generators given in Table C, where the asterisks represent elements of a form not allowed in the original set of generators --e.g. a generator of the form $\dot{y}^2 \partial_x$. The values for the asterisked commutators are given in Appendix A.

As is demonstrated by example in Appendix A, additional generators involving higher powers of \dot{y} can be obtained through the operation of commutation. But even if the set is augmented by the additional generators produced by commutation, the augmented set does not close under a finite number of commutations. The proof of this fact is given below.

Let X_n'' be a generator of the form

$$X_n'' = A \dot{y}^n \partial_x \quad \text{where } A = \text{const.} \neq 0, n > 0$$

Take the commutator of U_{12}'' with X_n''

$$[U_{12}'', X_n''] = [x \dot{y} \partial_x - \dot{y}^2 \partial_{\dot{y}}, A \dot{y}^n \partial_x] = -A(n+1) \dot{y}^{(n+1)} \partial_x$$

The commutator is thus a new generator of the same form as X_n'' but of a higher power of \dot{y} ; in fact the commutator is X_{n+1}'' . Thus the m^{th} commutator of U_{12}'' with X_n'' , i.e.,

$$\underbrace{[U_{12}'', [U_{12}'', \dots [U_{12}'', X_n''] \dots]]}_{m \text{ times}}$$

will produce an operator of the form $B \dot{y}^{(n+m)} \partial_x$. Now U_{10}'' is of the form of X_1'' (where $A=1$). Therefore if one takes the commutator of U_{12}'' with U_{10}'' and then the commutator of U_{12}'' with the result of the former operation, and so on, one obtains an infinite set of generators. If one allows the set to include powers of \dot{y} up to n , the n^{th} commutator of U_{12}'' with U_{10}'' will produce a generator not allowed in the set, namely a generator of the form $C \dot{y}^{(n+1)} \partial_x$. Therefore it has been shown that the generators of $\dot{y}=0$ of the form given in equation (4-2)

do not close under a finite number of commutations.

Table E gives the solutions generated by the seven generators in which ξ and η^y involve y for this example. In the table C_1 and C_2 are arbitrary constants.

TABLE E

Generators	Solutions
$U_9'' - U_8''$	$y = C_1 x$
U_{10}''	$y = C_2$
U_{11}''	$y = C_2$
U_{12}''	$y = C_2$
U_{13}''	$y = C_2, y = C_1 x$
U_{14}''	$y = C_1 x$
U_{15}''	$y = C_1 x$

As the table shows, in this case every one of the generators yields some subclass of solutions.

SECTION 5: Harmonic Oscillator

$$\ddot{y} = -ky \quad (k > 0)$$

$$\xi = h_0(x, y) + h_1(x, y) \dot{y}$$

$$\eta^y = l_0(x, y)$$

From the equation given on page 9 we obtain for $f = -ky$, $m=0$, $n=1$

$$4''(\ddot{y} + ky) \Big|_{\ddot{y} = ky} = (l_{0xx} - l_{0y} ky + 2h_{0x} ky - 2h_1 k^2 y^2 + l_0 k) \\ + (2l_{0xy} - h_{0xx} + 4h_{1x} ky + 3h_{0y} ky) \dot{y} + (l_{0yy} - 2h_{0xy} - h_{1xx} + 5h_{1y} ky \\ + h_1 k) \dot{y}^2 + (-h_{0yy} - 2h_{1xy}) \dot{y}^3 + (-h_{1yy}) \dot{y}^4 = 0$$

Since \dot{y} is arbitrary we obtain the following system of determining equations

$$l_{0xx} - l_{0y} ky + 2h_{0x} ky - 2h_1 k^2 y^2 + l_0 k = 0 \quad (5-1)$$

$$2l_{0xy} - h_{0xx} + 4h_{1x} ky + 3h_{0y} ky = 0 \quad (5-2)$$

$$l_{0yy} - 2h_{0xy} - h_{1xx} + 5h_{1y} ky + h_1 k = 0 \quad (5-3)$$

$$-h_{0yy} - 2h_{1xy} = 0 \quad (5-4)$$

$$-h_{1yy} = 0 \quad (5-5)$$

$$(5-5) \Rightarrow h_1 = [\alpha(x)]y + \beta(x)$$

$$(5-4) \Rightarrow h_0 = [-\alpha_x]y^2 + [\gamma(x)]y + \delta(x)$$

$$(5-3) \Rightarrow l_{0yy} = (-3\alpha_{xx} - 6\alpha k)y + (2\gamma_x + \beta_{xx} - \beta k)$$

Therefore $l_0 = \left[-\frac{\alpha_{xx}}{2} - \alpha k\right]y^3 + \left[\gamma_x + \frac{\beta_{xx}}{2} - \frac{\beta k}{2}\right]y^2 + [\lambda(x)]y + \mu(x)$

$$(5-2) \Rightarrow (-2\alpha_{xxx} - 8k\alpha_x)y^2 + (3\gamma_{xx} + 3k\gamma + 2\beta_{xxx} + 2k\beta_x)y + (2\lambda_x - \delta_{xxx}) = 0$$

Thus since y is arbitrary

$$2\alpha_{xxx} + 8k\alpha_x = 0 \Rightarrow \alpha = A_1 e^{\sqrt{-4k}x} + A_2 e^{-\sqrt{-4k}x} + a$$

$$3\gamma_{xx} + 3k\gamma + 2\beta_{xxx} + 2k\beta_x = 0 \quad (5-6)$$

$$2\lambda_x - \delta_{xxx} = 0 \quad (5-7)$$

$$(5-1) \Rightarrow \left(-\frac{\alpha_{xxx}}{2} - 2k\alpha_{xx}\right)y^3 + \left(\gamma_{xxx} + \frac{\beta_{xxx}}{2} - k\beta_{xx} + k\gamma_x - \frac{3}{2}k^2\beta\right)y^2 \\ + (\lambda_{xx} + 2k\delta_x)y + (\mu_{xx} + k\mu) = 0$$

Therefore since y is arbitrary

$$\frac{\alpha_{xxx}}{2} + 2k\alpha_{xx} = 0$$

$$\gamma_{xxx} + k\gamma_x + \frac{\beta_{xxx}}{2} - k\beta_{xx} - \frac{3}{2}k^2\beta = 0 \quad (5-8)$$

$$\lambda_{xx} + 2k\delta_x = 0 \quad (5-9)$$

$$\mu_{xx} + k\mu = 0 \Rightarrow \mu = B_1 e^{\sqrt{-k}x} + B_2 e^{-\sqrt{-k}x}$$

$$(5-9), (5-7) \Rightarrow \delta_{xxx} + 4k\delta_x = 0 \Rightarrow \delta = E_1 e^{\sqrt{-4k}x} + E_2 e^{-\sqrt{-4k}x} + g$$

$$(5-7) \Rightarrow \lambda = \sqrt{-k} E_1 e^{\sqrt{-4k}x} - \sqrt{-k} E_2 e^{-\sqrt{-4k}x} + c$$

$$(5-8), (5-6) \Rightarrow \beta_{xxx} + 10k\beta_{xx} + 9k^2\beta = 0 \Rightarrow \beta = D_1 e^{\sqrt{-k}x} + D_2 e^{-\sqrt{-k}x} + D_3 e^{\sqrt{-9k}x} + D_4 e^{-\sqrt{-9k}x}$$

$$(5-3) \Rightarrow \gamma_{xxx} + k\gamma_x + 48k^2 D_3 e^{\sqrt{-9k}x} + 48k^2 D_4 e^{-\sqrt{-9k}x} = 0$$

Therefore

$$\gamma = G_1 e^{\sqrt{-k}x} + G_2 e^{-\sqrt{-k}x} - \sqrt{-4k} D_3 e^{\sqrt{-9k}x} + \sqrt{-4k} D_4 e^{-\sqrt{-9k}x} + j$$

$$(5-6) \Rightarrow 3\gamma_{xx} + 3k\gamma - 16k\sqrt{-9k} D_3 e^{\sqrt{-9k}x} + 16k\sqrt{-9k} D_4 e^{-\sqrt{-9k}x} = (-3k + 3k)G_1 e^{\sqrt{-k}x} + (-3k + 3k)G_2 e^{-\sqrt{-k}x} + (27k\sqrt{-4k} - 3k\sqrt{-4k} - 16k\sqrt{-9k})D_3 e^{\sqrt{-9k}x} + (-27k\sqrt{-4k} + 3k\sqrt{-4k} + 16k\sqrt{-9k})D_4 e^{-\sqrt{-9k}x} = 0 \Rightarrow j = 0$$

Hence we have

$$U = A_1 e^{\sqrt{-4k}x} \left[(-\sqrt{-4k}y^2 + yj) \partial_x + (ky^3) \partial_y \right] + A_2 e^{-\sqrt{-4k}x} \left[(\sqrt{-4k}y^2 + yj) \partial_x + (ky^3) \partial_y \right] + a \left[(yj) \partial_x + (-ky^3) \partial_y \right] + G_1 e^{\sqrt{-k}x} \left[(y) \partial_x + (\sqrt{-k}y^2) \partial_y \right] + G_2 e^{-\sqrt{-k}x} \left[(y) \partial_x + (-\sqrt{-k}y^2) \partial_y \right] + g \left[\partial_x \right] + c \left[y \partial_y \right] + D_1 e^{\sqrt{-k}x} \left[(ij) \partial_x + (-ky^2) \partial_y \right] + D_2 e^{-\sqrt{-k}x} \left[(ij) \partial_x + (-ky^2) \partial_y \right] + D_3 e^{\sqrt{-9k}x} \left[(-\sqrt{-4k}y + j) \partial_x + (ky^2) \partial_y \right] + D_4 e^{-\sqrt{-9k}x} \left[(\sqrt{-4k}y + j) \partial_x + (ky^2) \partial_y \right] + E_1 e^{\sqrt{-4k}x} \left[\partial_x + (\sqrt{-k}y) \partial_y \right] + E_2 e^{-\sqrt{-4k}x} \left[\partial_x + (-\sqrt{-k}y) \partial_y \right] + B_1 e^{\sqrt{-k}x} \left[\partial_y \right] + B_2 e^{-\sqrt{-k}x} \left[\partial_y \right]$$

The extended operators are listed below

$$X_1'' = e^{\sqrt{-4k}x} \left[(-\sqrt{-4k}y^2 + yj) \partial_x + ky^3 \partial_y + (k\sqrt{-4k}y^3 + \sqrt{-4k}y^2j - y^3) \partial_y - k^2y^3 \partial_y \right]$$

$$X_2'' = e^{-\sqrt{-4k}x} \left[(\sqrt{-4k}y^2 + yj) \partial_x + ky^3 \partial_y + (-k\sqrt{-4k}y^3 - \sqrt{-4k}y^2j - y^3) \partial_y - k^2y^3 \partial_y \right]$$

$$X_3'' = yj \partial_x - ky^3 \partial_y + (-2ky^2j - y^3) \partial_y + k^2y^3 \partial_y$$

$$X_4'' = e^{\sqrt{-k}x} \left[y \partial_x + \sqrt{-k}y^2 \partial_y + (-ky^2 + \sqrt{-k}y^2j - y^2) \partial_y - k\sqrt{-k}y^2 \partial_y \right]$$

$$X_5'' = e^{-\sqrt{-k}x} \left[y \partial_x - \sqrt{-k}y^2 \partial_y + (-ky^2 - \sqrt{-k}y^2j - y^2) \partial_y + k\sqrt{-k}y^2 \partial_y \right]$$

$$X_6'' = \partial_x$$

$$X_7'' = y \partial_y + yj \partial_y + y^2 \partial_y$$

$$X_8'' = e^{\sqrt{-k}x} \left[y \partial_x - ky^2 \partial_y + (-k\sqrt{-k}y^2 - ky^2j - \sqrt{-k}y^2) \partial_y + k^2y^2 \partial_y \right]$$

$$\begin{aligned}
 X_9'' &= e^{-\sqrt{k}x} \left[\dot{y} \partial_x - k y^2 \partial_y + (k \sqrt{k} y^2 - k y \dot{y} + \sqrt{k} \dot{y}^2) \partial_{\dot{y}} + k^2 y^2 \partial_{\ddot{y}} \right] \\
 X_{10}'' &= e^{\sqrt{k}x} \left[(-\sqrt{k} y + \dot{y}) \partial_x + k y^2 \partial_y + (k \sqrt{k} y^2 - 3 k y \dot{y} - \sqrt{k} \dot{y}^2) \partial_{\dot{y}} - k^2 y^2 \partial_{\ddot{y}} \right] \\
 X_{11}'' &= e^{-\sqrt{k}x} \left[(\sqrt{k} y + \dot{y}) \partial_x + k y^2 \partial_y + (-k \sqrt{k} y^2 - 3 k y \dot{y} + \sqrt{k} \dot{y}^2) \partial_{\dot{y}} - k^2 y^2 \partial_{\ddot{y}} \right] \\
 X_{12}'' &= e^{\sqrt{k}x} \left[\partial_x + \sqrt{k} y \partial_y + (-2 k y - \sqrt{k} \dot{y}) \partial_{\dot{y}} - k \sqrt{k} y \partial_{\ddot{y}} \right] \\
 X_{13}'' &= e^{-\sqrt{k}x} \left[\partial_x - \sqrt{k} y \partial_y + (-2 k y + \sqrt{k} \dot{y}) \partial_{\dot{y}} + k \sqrt{k} y \partial_{\ddot{y}} \right] \\
 X_{14}'' &= e^{\sqrt{k}x} \left[\partial_y + \sqrt{k} \partial_{\dot{y}} - k \partial_{\ddot{y}} \right] \\
 X_{15}'' &= e^{-\sqrt{k}x} \left[\partial_y - \sqrt{k} \partial_{\dot{y}} - k \partial_{\ddot{y}} \right]
 \end{aligned}$$

An equivalent way of writing this set of operators is given in Table F.

TABLE F

U_1''	$\sin \sqrt{k}x \partial_y + \sqrt{k} \cos \sqrt{k}x \partial_{\dot{y}} - k \sin \sqrt{k}x \partial_{\ddot{y}}$
U_2''	$\cos \sqrt{k}x \partial_y - \sqrt{k} \sin \sqrt{k}x \partial_{\dot{y}} - k \cos \sqrt{k}x \partial_{\ddot{y}}$
U_3''	∂_x
U_4''	$y \partial_y + \dot{y} \partial_{\dot{y}} - k y \partial_{\ddot{y}}$
U_5''	$\sin 2\sqrt{k}x \partial_x + \sqrt{k} y \cos 2\sqrt{k}x \partial_y + (\dot{y} \sqrt{k} \cos 2\sqrt{k}x - 2 k y \sin 2\sqrt{k}x) \partial_{\dot{y}} - k^{3/2} y \cos 2\sqrt{k}x \partial_{\ddot{y}}$
U_6''	$\cos 2\sqrt{k}x \partial_x - \sqrt{k} y \sin 2\sqrt{k}x \partial_y + (\dot{y} \sqrt{k} \sin 2\sqrt{k}x - 2 k y \cos 2\sqrt{k}x) \partial_{\dot{y}} + k^{3/2} y \sin 2\sqrt{k}x \partial_{\ddot{y}}$
U_7''	$y \cos \sqrt{k}x \partial_x - \sqrt{k} y^2 \sin \sqrt{k}x \partial_y + (-k y^2 \cos \sqrt{k}x - \dot{y}^2 \cos \sqrt{k}x - \sqrt{k} y \dot{y} \sin \sqrt{k}x) \partial_{\dot{y}} + k^{3/2} y^2 \sin \sqrt{k}x \partial_{\ddot{y}}$
U_8''	$y \sin \sqrt{k}x \partial_x + \sqrt{k} y^2 \cos \sqrt{k}x \partial_y + (-k y^2 \sin \sqrt{k}x - \dot{y}^2 \sin \sqrt{k}x + \sqrt{k} y \dot{y} \cos \sqrt{k}x) \partial_{\dot{y}} - k^{3/2} y^2 \cos \sqrt{k}x \partial_{\ddot{y}}$

TABLE F (continuation)

U_9''	$(2\sqrt{k} y^2 \sin 2\sqrt{k}x + y \dot{y} \cos 2\sqrt{k}x) \partial_x + k y^3 \cos 2\sqrt{k}x \partial_y + (-2k^{3/2} y^3 \sin 2\sqrt{k}x - 2\sqrt{k} y \dot{y}^2 \sin 2\sqrt{k}x - \dot{y}^3 \cos 2\sqrt{k}x) \partial_{\dot{y}} - k^2 y^3 \cos 2\sqrt{k}x \partial_{\ddot{y}}$
U_{10}''	$(-2\sqrt{k} y^2 \cos 2\sqrt{k}x + y \dot{y} \sin 2\sqrt{k}x) \partial_x + k y^3 \sin 2\sqrt{k}x \partial_y + (2k^{3/2} y^3 \cos 2\sqrt{k}x + 2\sqrt{k} y \dot{y}^2 \cos 2\sqrt{k}x - \dot{y}^3 \sin 2\sqrt{k}x) \partial_{\dot{y}} - k^2 y^3 \sin 2\sqrt{k}x \partial_{\ddot{y}}$
U_{11}''	$\dot{y} \cos \sqrt{k}x \partial_x - k y^2 \cos \sqrt{k}x \partial_y + (-k y \dot{y} \cos \sqrt{k}x + k^{3/2} y^2 \sin \sqrt{k}x + \sqrt{k} \dot{y}^2 \sin \sqrt{k}x) \partial_{\dot{y}} + k^2 y^2 \cos \sqrt{k}x \partial_{\ddot{y}}$
U_{12}''	$\dot{y} \sin \sqrt{k}x \partial_x - k y^2 \sin \sqrt{k}x \partial_y + (-k y \dot{y} \sin \sqrt{k}x - k^{3/2} y^2 \cos \sqrt{k}x - \sqrt{k} \dot{y}^2 \cos \sqrt{k}x) \partial_{\dot{y}} + k^2 y^2 \sin \sqrt{k}x \partial_{\ddot{y}}$
U_{13}''	$(2\sqrt{k} y \sin 3\sqrt{k}x + \dot{y} \cos 3\sqrt{k}x) \partial_x + k y^2 \cos 3\sqrt{k}x \partial_y + (-3k^{3/2} y^2 \sin 3\sqrt{k}x - 3k y \dot{y} \cos 3\sqrt{k}x + \sqrt{k} \dot{y}^2 \sin 3\sqrt{k}x) \partial_{\dot{y}} - k^2 y^2 \cos 3\sqrt{k}x \partial_{\ddot{y}}$
U_{14}''	$(-2\sqrt{k} y \cos 3\sqrt{k}x + \dot{y} \sin 3\sqrt{k}x) \partial_x + k y^2 \sin 3\sqrt{k}x \partial_y + (3k^{3/2} y^2 \cos 3\sqrt{k}x - 3k y \dot{y} \sin 3\sqrt{k}x - \sqrt{k} \dot{y}^2 \cos 3\sqrt{k}x) \partial_{\dot{y}} - k^2 y^2 \sin 3\sqrt{k}x \partial_{\ddot{y}}$
U_{15}''	$y \dot{y} \partial_x - k y^3 \partial_y + (-2k y^2 \dot{y} - \dot{y}^3) \partial_{\dot{y}} + k^2 y^3 \partial_{\ddot{y}}$

The first eight generators listed in Table F are Lie operators that have not previously been reported.* The rest are new AKW generators.

The commutators for the generators listed in Table F are given in Table G. The asterisks represent elements of a form not allowed in the original set of generators. The values for the asterisked commutators

* These eight generators were derived by Robert L. Anderson in his notes for the course on the Group Theoretic Properties of Differential Equations previously cited.

are given in Appendix B.* In the table $K=\sqrt{k}$.

TABLE G

$[U_i, \rightarrow]$	U_1''	U_2''	U_3''	U_4''	U_5''	U_6''	U_7''	U_8''	U_9''	U_{10}''	U_{11}''	U_{12}''	U_{13}''	U_{14}''	U_{15}''
U_1''		0	$-KU_4''$	U_1''	$-KU_1''$	$-KU_2''$	$-\frac{1}{2}KU_3'' + \frac{1}{2}U_5''$	$\frac{1}{2}U_3'' - \frac{1}{2}U_6''$	*	*	*	*	*	*	*
U_2''			KU_1''	U_2''	KU_2''	$-KU_1''$	$\frac{1}{2}U_3'' + \frac{1}{2}U_6''$	$\frac{3}{2}KU_4'' + \frac{1}{2}U_5''$	*	*	*	*	*	*	*
U_3''				0	$2KU_6''$	$-2KU_5''$	$-KU_8''$	KU_7''	$2KU_{10}''$	$2KU_9''$	$-KU_{12}''$	KU_{11}''	$-3KU_{14}''$	$3KU_{13}''$	0
U_4''					0	0	U_7''	U_8''	$2U_9''$	$2U_{10}''$	U_{11}''	U_{12}''	U_{13}''	U_{14}''	$2U_{15}''$
U_5''						$-2KU_3''$	$-KU_4''$	KU_8''	*	*	*	*	*	*	*
U_6''							KU_8''	KU_7''	*	*	*	*	*	*	*
U_7''								0	*	*	*	*	*	*	*
U_8''									*	*	*	*	*	*	*
U_9''										*	*	*	*	*	*
U_{10}''											*	*	*	*	*
U_{11}''												*	*	*	*
U_{12}''													*	*	*
U_{13}''														*	*
U_{14}''															*
U_{15}''															

It can be shown that these generators do not close under a finite

* The commutators of the generators listed in Table F were run by David Davison with his computer program Mando (see his M.Sc. thesis, Department of Physics, University of the Pacific, 1973).

number of commutations. The proof is given below.

Consider

$$R'' = y\dot{y} \partial_x - k y^3 \partial_y + (-2k y^2 \dot{y} - \dot{y}^3) \partial_{\dot{y}} + k^2 y^3 \partial_{\ddot{y}}$$

$$X'' = A y^{a_1} \dot{y}^{a_2} \partial_x + B y^{b_1} \dot{y}^{b_2} \partial_y + C y^{c_1} \dot{y}^{c_2} \partial_{\dot{y}} + E y^{e_1} \dot{y}^{e_2} \partial_{\ddot{y}}$$

where the capital letters are infinitely differentiable functions of x ,

the letters with index 1 are not zero and neither b_1+1+c_1 nor b_2+e_2+2 .

Therefore $[R'', X''] = A_x y^{(a_1+1)} \dot{y}^{(a_2+1)} \partial_x + B_x y^{(b_1+1)} \dot{y}^{(b_2+1)} \partial_y + C_x y^{(c_1+1)} \dot{y}^{(c_2+1)} \partial_{\dot{y}}$
 $- a_2 E y^{e_1} \dot{y}^{(e_2+2)} \partial_{\ddot{y}} - B_y b_1 \dot{y}^{(b_2+1)} \partial_x - C_y c_1 \dot{y}^{c_2} \partial_x + E_x y^{(e_1+1)} \dot{y}^{(e_2+1)} \partial_{\dot{y}}$
 $- a_1 k A_y y^{(a_1+2)} \dot{y}^{a_2} \partial_x - b_1 k B_y y^{(b_1+2)} \dot{y}^{b_2} \partial_y + 3k B_y y^{(b_1+2)} \dot{y}^{b_2} \partial_y - k c_1 C_y y^{(c_1+2)} \dot{y}^{c_2} \partial_{\dot{y}}$
 $- k e_1 E_y y^{(e_1+2)} \dot{y}^{e_2} \partial_{\ddot{y}} - 2k a_2 A_y y^{(a_1+2)} \dot{y}^{a_2} \partial_x - 2b_2 k B_y y^{(b_1+2)} \dot{y}^{b_2} \partial_y + 2k B_y y^{(b_1+2)} \dot{y}^{b_2} \partial_{\dot{y}}$
 $- 2c_2 k C_y y^{(c_1+2)} \dot{y}^{c_2} \partial_{\dot{y}} + 2k C_y y^{(c_1+2)} \dot{y}^{c_2} \partial_{\ddot{y}} - 2e_2 k E_y y^{(e_1+2)} \dot{y}^{e_2} \partial_{\ddot{y}} - a_2 A_y y^{a_1} \dot{y}^{(a_2+2)} \partial_x$
 $- b_2 B_y y^{b_1} \dot{y}^{(b_2+2)} \partial_y - c_2 C_y y^{c_1} \dot{y}^{(c_2+2)} \partial_{\dot{y}} + 3C_y c_1 \dot{y}^{(c_2+2)} \partial_{\ddot{y}} - 3k^2 B_y y^{(b_1+2)} \dot{y}^{b_2} \partial_{\ddot{y}}$

Thus we recover terms of the form of X'' plus terms which cannot cancel them. Therefore when we take the commutators $[R'', X'']$, $[R'', [R'', X'']]$, etc. we obtain new generators of the same form as X'' with higher powers of \dot{y} . Therefore a set of generators which includes operators of the form of R'' and X'' cannot close. But $U_{15}'' = R''$ and U_7'' and U_8'' are of the form of X'' . Therefore the generators listed in Table F cannot close under a finite number of commutations.

The solutions generated by the seven non-Lie generators of the harmonic oscillator are given in Table H. In this case only three of the generators yield non-trivial solutions and each of those yield the general solution. In the table A_1 and A_2 are arbitrary constants.

TABLE H

Generators	Solutions
u''_9	$y=0$
u''_{10}	$y=0$
u''_{11}	$y = A_{\frac{1}{2}} e^{\pm \sqrt{-\frac{1}{2}} x}$
u''_{12}	$y = A_{\frac{1}{2}} e^{\pm \sqrt{-\frac{1}{2}} x}$
u''_{13}	$y=0$
u''_{14}	$y=0$
u''_{15}	$y = A_{\frac{1}{2}} e^{\pm \sqrt{-\frac{1}{2}} x}$

CHAPTER IV : Counting Theorems for Ordinary Differential
Equations of Order Higher than Two

SECTION 1: Lie's Counting Theorem for Ordinary Differential Equations
of R^{th} Order ($R > 2$)* 22

Sophus Lie published a counting theorem for ordinary differential equations of order higher than two in his book "Vorlesungen Über Continuerliche Gruppen" in 1893. The new counting theorem given in Section 2 of this chapter is based on Lie's version although there are several basic differences. For this reason Lie's theorem is given here in some detail.

Lemma: Given a differential equation of order r ($r > 2$)

$$y^{(r)} - f(x, y, \dot{y}, \dots, y^{(r-1)}) = 0$$

it is always possible to define a region in the xy -plane in which always one and only one integral curve passes through two points chosen in the region such that at one of the two points prescribed values for $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ are given.

Theorem: Given a differential equation of order r ($r > 2$) which is integrable "in the large"

* I would like to thank David Davison for translating Lie's theorem and the lemma preceding it.

$$y^{(r)} - f(x, y, \dot{y}, \dots, y^{(r-1)}) = 0 \quad (1-1)$$

there exist at most $(r+4)$ linearly independent infinitesimal generators of the form

$$U^{(r)} = \xi \partial_x + \eta^y \partial_y + \eta^{\dot{y}} \partial_{\dot{y}} + \dots + \eta^{y^{(r)}} \partial_{y^{(r)}} \quad (1-2)$$

where

$$\xi = \xi(x, y), \quad \eta^y = \eta^y(x, y) \quad (1-3)$$

which leave the differential equation covariant.

Proof: Assuming that equation (1-1) admits \mathfrak{f} independent infinitesimal transformations, then it also admits those of the form

$$U \equiv a_1 U_1 + a_2 U_2 + \dots + a_{\mathfrak{f}} U_{\mathfrak{f}}$$

Of these transformations there are at least $\mathfrak{f}-4$ independent ones which leave two arbitrarily chosen points p, q invariant within the plane, as the invariance of a point is expressed by at most two relations

$$\begin{aligned} \sum_{k=1}^{\mathfrak{f}} a_k \xi_k(x_j, y_j) &= 0 \\ \sum_{k=1}^{\mathfrak{f}} a_k \eta_k^y(x_j, y_j) &= 0 \end{aligned} \quad (1-4)$$

Through both p and q pass exactly an $(r-2)$ -fold infinity of integral curves, since the solution of an equation of r^{th} order contains r arbitrary constants and two of them ---that is, two points--- have already been fixed. An alternate way of stating this is that by the lemma we need two points and $(r-2)$ derivatives at one of the points to fix a curve between those points; since the points have been fixed but the derivatives have not, we still have $(r-2)$ degrees of freedom left. Therefore, in order to fix a curve between the two points we must fix $(r-2)$ derivatives at one of the points, i.e., we must set

$$\begin{aligned}\sum_{k=1}^{p-4} a_k \eta_k^{\dot{y}}(x, y, \dot{y}) &= 0 \\ \sum_{k=1}^{p-4} a_k \eta_k^{\ddot{y}}(x, y, \dot{y}, \ddot{y}) &= 0 \\ &\vdots \\ \sum_{k=1}^{p-4} a_k \eta_k^{y^{(r-2)}}(x, y, \dot{y}, \dots, y^{(r-2)}) &= 0\end{aligned}$$

Note that since ξ and η^y depend only on x and y in Lie's case and since $\eta^{y^{(q)}}$ depends at most on $y^{(q)}$ — that is, no higher derivatives than q are included in $\eta^{y^{(q)}}$. Therefore none of the $\eta^{y^{(q)}}$ up to and including $\eta^{y^{(r-2)}}$ depends on $y^{(r-1)}$. Hence, since one needs at most $(r-2)$ equations to fix a curve, one is left with at least $\sigma = p-4-(r-2)$ independent infinitesimal transformations of (1-1) which leave (1-1) covariant and p, q and an integral curve c through p, q invariant.

There are ∞^1 integral curves which pass through p and have the same values for $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at p . The independent generators transform these ∞^1 integral curves among themselves since they leave p and the values $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ of c at p unchanged and since, as mentioned above, $\eta^{y^{(q)}}$ ($q \leq (r-2)$) does not depend on $y^{(r-1)}$, the only derivative which has not yet been fixed. In order to fix one of these curves (say \mathcal{W}) we must set

$$\sum_{k=1}^{\sigma} a_{1k} \eta_{\mathcal{W}}^{y^{(r-1)}} = 0$$

at p . Therefore there are at least $\sigma-1$ independent infinitesimal transformations which leave equation (1-1) covariant and which, in addition to p, q and c , leave an integral curve \mathcal{W} (which passes through p and has the same values for $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c there) invariant.

Likewise we conclude that there are at least $\sigma-2$ (that is, $p-4-(r-2)-2$)

$= p-r-4$) independent infinitesimal transformations which leave (1-1) covariant and which, besides p, q, c and Π , leave yet another integral curve γ (which passes through q and has the same values for $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c there) invariant.

Since Π and c are fixed we must have

$$\sum_{k=1}^{r-2} a_k \eta_{c,k} y^{(r-1)} = 0$$

$$\sum_{k=1}^{r-2} a_k \eta_{\Pi,k} y^{(r-1)} = 0$$

But Lie has proved that $\eta_{y^{(q)}}$ is linear in $y^{(q)}$ for his case (where $q > 1$).²³ Therefore we obtain two equations of the form

$$\eta_i y^{(r-1)} = b_1 + b_2 y_i^{(r-1)} \quad (i = c, \Pi)$$

b_1 and b_2 are constants which apply both to c and Π because they depend only on $x, y, \dot{y}, \ddot{y}, \dots, y^{(r-2)}$ all of which have been fixed and are equal for both curves.

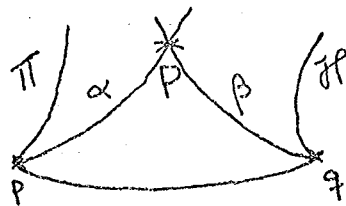
The determinant of the coefficients

$$\begin{vmatrix} y_c^{(r-1)} & 1 \\ y_{\Pi}^{(r-1)} & 1 \end{vmatrix} = (y_c^{(r-1)} - y_{\Pi}^{(r-1)})$$

is not equal to zero because c and Π are distinctly different curves and therefore $y_c^{(r-1)} \neq y_{\Pi}^{(r-1)}$. Therefore we have that the b 's are trivial, i.e.,

$$b_1 = b_2 = 0$$

which means that $\eta_{y^{(r-1)}} = 0$ for all integral curves through p with the same $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at p . Similarly one can prove that $\eta_{y^{(r-1)}} = 0$ for all integral curves through q which have the same $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at q . Therefore every value of $y^{(r-2)}$ at points p and q remains invariant as long as the values set for $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ are the same as those set for c at p and q , respectively.



Now we choose a point within the region but not on the curve c .

By the lemma, an integral curve α (which has the same values for $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at p) through p and an integral curve β (which similarly has the same values of $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at q) through q pass through P . Since our $r-1$ independent transformations (call them X) leave every integral curve (which has the same values of $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at p and q , respectively) invariant, they must leave α , β and the intersection P of α and β invariant. All points of the region and therefore all points of the entire plane, with the possible exception of points on c , remain invariant under the action of X .

In the original theorem Lie never mentioned that if P lies on c the two invariant curves through P are actually one and the same curve c . Note that this is the only case in which this can happen because there is only one curve, namely c , with the same $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as c at p that passes through both p and q . This problem, however, can be solved as follows. Once we have found that P and all points on the plane except those points which are part of c are invariant, we can find an integral curve between P and Q , where Q is an arbitrary point on c ; that such a curve, γ , exists and is unique is given by the lemma. Since all points, with the possible exception of Q , on this curve are invariant, $\dot{y}, \ddot{y}, \dots, y^{(r-1)}$ at P are also invariant. Therefore the curve γ is invariant, since r constants define an integral curve of an r^{th} order equation. Since Q

is allowed to move along c but not along \mathcal{C} , Q must be invariant. But Q is an arbitrary point on c ; therefore all points on c must be invariant. Hence the whole plane is invariant and no extra equations need be added to make all the points on c invariant.

All of this means that $X=0$. The maximum number of linearly independent infinitesimal transformations which can add up to zero is zero, since if this number were any greater the transformations would be linearly dependent. Therefore we have $\mathfrak{P}=r-4$ which means that $\mathfrak{P}=r-4$ is the maximum number of linearly independent infinitesimal transformations of the Lie form which leave an equation of r^{th} order ($r>2$) covariant.

Lie gives

$$y^{(r)} = 0 \quad (r > 2)$$

as an example of an equation of r^{th} order which produces the maximum allowed number of generators. He states²⁴ that these transformations are

$$\partial_x, \partial_y, x\partial_x, y\partial_y, x\partial_y, x^2\partial_y, \dots, x^{(r-1)}\partial_y, x^2\partial_x + (r-1)xy\partial_y$$

SECTION 2: Generalized Counting Theorem for Ordinary Differential Equations of Third Order

This theorem is a generalization of Lie's counting theorem for ordinary differential equations of order higher than the second. Above the generalized theorem for equations of second order (see page 22), it was explained that in order to generalize Lie it is necessary to assume a particular form for the y dependence of ξ and η^4 . Similarly, in the case of an r^{th} order equation it is necessary to assume a particular

$\dot{y}, \ddot{y}, \dots, y^{(r-1)}$ dependence for ξ and η^y . It is also necessary to assume a particular form for the $\dot{y}, \ddot{y}, \dots, y^{(r-1)}$ dependence of f , since as will be seen the coefficients of f are not always zero in this case.

For reasons explained below (see page 53), Lie's theorem cannot be generalized to apply to equations of r^{th} order ($r > 2$) without an equation which gives $\eta^{y^{(2)}}$ in terms of powers of $y^{(r-1)}$ for the general case. As such an equation is not known, I confine myself here to a generalized counting theorem for third order equations with an explanation as to how the theorem could be generalized to the r^{th} order case if the above mentioned equation for $\eta^{y^{(2)}}$ could be found. Such an explanation is equivalent to a formula for finding the maximum number of linearly independent infinitesimal generators which leave an equation of n^{th} order covariant provided ξ and η^y are of a given form and all the $\eta^{y^{(2)}}$ are found explicitly in terms of $y^{(n-1)}$.

Theorem: A third order ordinary differential equation which is integrable "in the large" and is of the form

$$\ddot{y}^3 - f(x, y, \dot{y}, \ddot{y}) = 0 \quad (2-1)$$

where $f = \sum_{t=0}^{\infty} g_t(x, y) \dot{y}^t \ddot{y}^3$ (2-2)

is covariant under the action of at most 3 linearly independent generators of the form

$$U_k = \xi_k \partial_x + \eta_k^y \partial_y + \eta_k^{\dot{y}} \partial_{\dot{y}} + \eta_k^{\ddot{y}} \partial_{\ddot{y}} + \eta_k^{\ddot{\ddot{y}}} \partial_{\ddot{\ddot{y}}} \quad (2-3)$$

where ξ_k and η_k^y are of the form

$$\begin{aligned} \xi_k &= \sum_{i,j=0}^{m_1} h_{ij,k}(x, y) \dot{y}^i \ddot{y}^j \\ \eta_k^y &= \sum_{i,j=0}^{m_2} h_{ij,k}(x, y) \dot{y}^i \ddot{y}^j \end{aligned} \quad (2-4)$$

subject to the condition that precisely one function for each non-trivial pair $\lambda_{ijk}, h_{(i-1)jk}$ ($i, j = 1, 2, \dots, m$ or n , and all k) is specified a priori (see page), where $\rho = 7$ if $m=n=0$ and $\rho = c+d+e+u+M^2+N^2+3M+3N+5$ if either m or n is not equal to zero and the maximum c, d, e and u are

given by

$$\begin{aligned} \text{max. } c & \begin{cases} \text{for } 0 \leq v \leq 2 & \begin{cases} n+1 & (\text{for } n \geq m, n \geq 0, m \geq 0) \\ m+1 & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \\ \text{for } v > 2 & \begin{cases} n-1+v & (\text{for } n \geq m, n > 0, m > 0) \\ m-1+v & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \end{cases} \\ \text{max. } d & \begin{cases} \text{for } v \text{ arb.} & \begin{cases} n+1 & (\text{for } n \geq m, n \geq 0, m \geq 0) \\ m+1 & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \end{cases} \\ \text{max. } e & \begin{cases} \text{for } 0 \leq v \leq 2 & \begin{cases} n+2 & (\text{for } n \geq m, n \geq 0, m \geq 0) \\ m+2 & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \\ \text{for } v > 2 & \begin{cases} n-2+2p & (\text{for } n \geq m, n > 0, m > 0) \\ m-2+2p & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \end{cases} \\ \text{max. } u & \begin{cases} \text{for } 0 \leq v \leq 2 & \begin{cases} n+2 & (\text{for } n \geq m, n \geq 0, m \geq 0) \\ m+2 & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \\ \text{for } v > 2 & \begin{cases} n-2+p & (\text{for } n \geq m, n > 0, m > 0) \\ m-2+p & (\text{for } n < m, n \geq 0, m > 0) \end{cases} \end{cases} \end{aligned}$$

Proof: Lie's theorem given in Section 1 applies to the case where $m=n=0$ and will not be repeated here. For all other cases, assume that equation (2-1) yields \mathfrak{f} (where \mathfrak{f} is as yet unknown) independent infinitesimal transformations of the form given in equations (2-3) and (2-4), then it also yields those of the form

The invariance of a point is expressed by the relations

$$\begin{aligned}\sum_{k=1}^n a_k \xi_k(x_s, y_s, \dot{y}_s, \ddot{y}_s) &= 0 \\ \sum_{k=1}^n a_k \eta_k^y(x_s, y_s, \dot{y}_s, \ddot{y}_s) &= 0\end{aligned}\quad (2-5)$$

The invariance of a \dot{y} is expressed by the equation

$$\sum_{k=1}^n a_k \eta_k^{\dot{y}}(x_s, y_s, \dot{y}_s, \ddot{y}_s) = 0 \quad (2-6)$$

Therefore the invariance of a point p_1 on those curves with a particular value of \dot{y} at p_1 , call it \dot{y}_{p_1} , and the invariance of \dot{y}_{p_1} are expressed by the equations

$$\sum_{k=1}^n a_k \sum_{i=0}^m h_{ijk}(x_{p_1}, y_{p_1}) \dot{y}_{p_1}^i = 0 \quad (j=0, 1, \dots, m) \quad (2-7)$$

$$\sum_{k=1}^n a_k \sum_{i=0}^m l_{ijk}(x_{p_1}, y_{p_1}) \dot{y}_{p_1}^i = 0 \quad (j=0, 1, \dots, m) \quad (2-8)$$

$$b_{ip_1}(x_{p_1}, y_{p_1}, \dot{y}_{p_1}) = 0 \quad (2-9)$$

The b_i 's are the coefficients of the powers of \ddot{y} (including the power zero) in the equation for $\sum a_k \eta_k^{\dot{y}}$ given on page 10.

Note that in some cases equations (2-7) and (2-8) eliminate some of the coefficient of \ddot{y} : such coefficients will be referred to as "forced zero coefficients," abbreviated f.z.c. When such a f.z.c. is found to be the coefficient of the highest power of \ddot{y} it will be called "highest forced zero coefficient," or h.f.z.c. Note that derivatives of h_{ij} or l_{ij} cannot be f.z.c. In equation (2-9), c is the highest power of \ddot{y} in $\eta^{\dot{y}}$ after all the h.f.z.c. have been eliminated.

The invariance of a second point, p_2 , for curves with arbitrary \dot{y} at p_2 is expressed by the relations

$$\sum_{k=1}^n a_k h_{ijk}(x_{p_2}, y_{p_2}) = 0 \quad (i, j=0, 1, \dots, m) \quad (2-10)$$

$$\sum_{k=1}^n a_k l_{ijk}(x_{p_2}, y_{p_2}) = 0 \quad (i, j=0, 1, \dots, m) \quad (2-11)$$

Since we now have two invariant points, p_1 and p_2 , and one invariant \dot{y} , \dot{y}_{p_1} , and since by the lemma given in Section 1 of this chapter there is

therefore one and only one integral curve between p_1 and p_2 with that \dot{y}_{p_1} at p_1 , that curve, call it α_0 , is invariant under the transformations given. Therefore, since we had \int independent generators which leave equation (2-1) covariant we need at most $(n+1)+(m+1)+(c+1)+(m+1)^2+(n+1)^2 = c+m^2+n^2+3m+3n+5$ equations to leave two points and the integral curve between them invariant, we have $\int -c-m^2-n^2-3m-3n-5$ independent infinitesimal transformations which leave equation (2-1) covariant and two points, the curve between them and the \dot{y} of all the curves with the same \dot{y} at p_1 as α_0 there invariant.

Since α_0 is invariant

$$\sum_{k=1}^d a_k \eta_{p_2 k}^{\dot{y}_{\alpha_0}} = 0$$

for $\dot{y}_{\alpha_0 p_2}$. Let d be the highest power of \dot{y} in $\eta_{p_2}^{\dot{y}}$ after the h.f.z.c. have been eliminated. Since equations (2-10) and (2-11) are different than (2-7) and (2-8), there may be a different number of h.f.z.c. in $\eta_{p_2}^{\dot{y}}$ than in $\eta_{p_1}^{\dot{y}}$. Hence, in general, $c \neq d$. Take d integral curves α_i ($i=1,2,\dots,d$) which pass through p_2 and have the same \dot{y} at p_2 as α_0 there. Therefore we obtain a system of $d+1$ equations

$$\sum_{k=1}^d a_k \eta_{p_2 k}^{\dot{y}_{\alpha_i}} = 0 \quad (i=0,1,\dots,d) \quad (2-12)$$

Let z_i ($i=0,1,\dots,d$) be the coefficients of the powers of \dot{y} in $\eta_{p_2}^{\dot{y}_{\alpha_i}}$ (see page 10) including the power zero. Each z_i is a constant for the whole system of equations since z_i depends only on x_{p_2} , y_{p_2} and \dot{y}_{p_2} , all of which have been fixed and are equal in all the equations (2-12). Therefore the determinant of the coefficients

$$\begin{vmatrix} \ddot{y}_{\alpha_0}^d & \dots & \ddot{y}_{\alpha_0}^1 & 1 \\ \vdots & & \vdots & \\ \ddot{y}_{\alpha_d}^d & \dots & \ddot{y}_{\alpha_d}^1 & 1 \end{vmatrix}$$

is a Vandermonde determinant (see page 13). Since the α_i 's are all distinctly different curves by choice, their \ddot{y} 's are all different and therefore the determinant given above cannot equal zero. Hence all the z_i 's are trivial, i.e.,

$$z_0 = z_1 = \dots = z_d = 0$$

Therefore $\eta_{p_2}^{\ddot{y}} = 0$ which means that the \dot{y} of any integral curve which passes through p_2 and has the same \dot{y} as α_0 at p_2 will remain invariant, regardless of the \ddot{y} of the curve. Thus we have added d more equations which relate the original generators, so we are left with $\rho - c - d - m^2 - n^2 - 3m - 3n - 5$ independent infinitesimal transformations which leave equation (2-1) covariant as well as leaving two points, the curve between them and \dot{y}_{p_1} and \dot{y}_{p_2} (for all curves that have $\dot{y} = \dot{y}_{\alpha_0}$ at p_1 and p_2 , respectively) invariant.

Since α_0 is invariant, its \ddot{y} is invariant at all points. Therefore

$$\sum_{k=1}^d \alpha_k \eta_{\alpha_0 k}^{\ddot{y}_{p_1}} = \sum_{k=1}^d \alpha_k \eta_{\alpha_0 k}^{\ddot{y}_{p_2}} = 0$$

Let e be the highest power of \ddot{y} in $\eta_{p_1}^{\ddot{y}}$ after the h.f.z.c. have been eliminated (see page 10). Let u be the highest power of \ddot{y} in $\eta_{p_2}^{\ddot{y}}$ after the h.f.z.c. have been eliminated. Note that in general equ. Set

$$\sum_{k=1}^d \alpha_k \eta_{\alpha_i k}^{\ddot{y}_{p_1}} = 0 \quad (j = u+1, \dots, u+e) \quad (2-13)$$

$$\sum_{k=1}^d \alpha_k \eta_{\alpha_i k}^{\ddot{y}_{p_2}} = 0 \quad (i = 1, \dots, u) \quad (2-14)$$

where the α_i 's are the same ones we saw before with a few added if $u > d$.

The α_j 's are new curves through p_1 which have the same \dot{y} as α_0 at p_1 . We therefore have two systems of equations, (2-13) and (2-14), each with its determinant

$$\begin{vmatrix} \ddot{y}_{\alpha_0} & \dots & \ddot{y}_{\alpha_0} & 1 \\ \ddot{y}_{\alpha_{(u+1)}} & \dots & \ddot{y}_{\alpha_{(u+1)}} & 1 \\ \vdots & & \vdots & \\ \ddot{y}_{\alpha_{(u+r)}} & \dots & \ddot{y}_{\alpha_{(u+r)}} & 1 \end{vmatrix}$$

$$\begin{vmatrix} \ddot{y}_{\alpha_0} & \dots & \ddot{y}_{\alpha_0} & 1 \\ \ddot{y}_{\alpha_1} & \dots & \ddot{y}_{\alpha_1} & 1 \\ \vdots & & \vdots & \\ \ddot{y}_{\alpha_u} & \dots & \ddot{y}_{\alpha_u} & 1 \end{vmatrix}$$

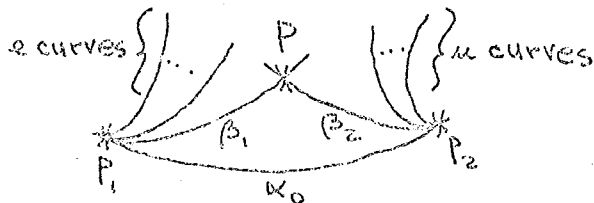
Both of these are Vandermonde determinants (see page 13). Again, since these are all distinct curves, their \ddot{y} 's must be different and therefore neither of the determinants can equal zero. Therefore the coefficients of the \ddot{y} 's must all be trivial. Thus

$$\sum_{k=1}^e a_k \gamma_{(\ddot{y}=\ddot{y}_{\alpha_0})_k}^{\ddot{y}_{P_1}} \equiv 0 \quad \sum_{k=1}^u a_k \gamma_{(\ddot{y}=\ddot{y}_{\alpha_0})_k}^{\ddot{y}_{P_2}} \equiv 0 \quad (2-15)$$

Since we have added $e+u$ equations to those already mentioned, we have

$\sigma = 9 - 6 - 4 - 6 - 2 - 2 - 3m - 3n - 5$ independent infinitesimal transformations

which besides leaving (2-1) covariant leave two points, the curve between them, e curves at p_1 and u curves at p_2 (all of which have the same \dot{y} as α_0 at p_1 and p_2 respectively) invariant since a transformation which leaves \dot{y} and \ddot{y} of an integral curve of a third order equation invariant at an invariant point must leave the curve invariant.



Now take a point P within the region, but not on α_0 . By the lemma we can find an integral curve through p_1 and P which has the same \dot{y} as α_0 at p_1 . We can also find an integral curve through p_2 and P which has

the same \dot{y} as α_0 at p_2 . Since these curves, call them β_1 and β_2 , are invariant by the above discussion, P is also invariant. Thus we have $G = f - c - d - e - u - m^2 - n^2 - 3m - 3n - 5$ independent transformations, call them X , which leave every point on the plane, with the possible exception of points on α_0 , invariant.

Once we have found that P and all points on the plane except those points which are part of α_0 are invariant, we can find an integral curve between P and Q , where Q is an arbitrary point on α_0 ; that such a curve, γ , exists and is unique is given by the lemma. Since all points, with the possible exception of Q , on this curve are invariant, $\dot{y}, \ddot{y}, \dots, y^{(r-1)}$ at P are also invariant. Therefore the curve γ is invariant, since r constants define an integral curve of an r^{th} order equation. Since Q is allowed to move along α_0 but not along γ , Q must be invariant. But Q is an arbitrary point on α_0 ; therefore all points on α_0 must be invariant. Hence the whole plane is invariant and no extra equations need be added to make all the points on α_0 invariant.

Thus $X \neq 0$. Now the maximum number of independent generators which can add up to zero is zero, since if this number were larger the generators would be linearly dependent. Hence $G = 0$, which means that the maximum number of linearly independent infinitesimal transformations of the form given in equations (2-3) and (2-4) which leave an equation of the form given in equations (2-1) and (2-2) covariant is $f = c + d + e + u + m^2 + n^2 + 3m + 3n + 5$ where the maximum values for c , d , e and u are determined by the equations on pages 10 and 48, and are listed on page 47.

The theorem given above can be extended to include any r^{th} order ordinary differential equation ($r > 2$) as long as the $y^{(r-1)}$ dependence of all the $\eta^{(q)}$ ($q=1,2,\dots,(r-1)$) is explicitly known. There is no equation known which will give $\eta^{(q)}$ directly in terms of powers of $y^{(r-1)}$ for arbitrary q . However the equation

$$\eta^{(q)} = \frac{d\eta^{(q-1)}}{dx} - y^{(q-1)} \frac{d\eta^{(q-1)}}{dy^{(r-1)}}$$

will give $\eta^{(q)}$ in terms of $y^{(r-1)}$ for any particular case. The extension is explained below. In Section 1 Lie's case was given for r^{th} order and will not be repeated here. In what follows it is assumed that either m or n is not zero.

Given an equation of the form

$$y^{(r)} - f(x, y, \dot{y}, \dots, y^{(r-1)}) = 0$$

where $f = \sum_{i_1, \dots, i_{r-1}} g(x, y) \dot{y}^{i_1} \ddot{y}^{i_2} \dots (y^{(r-1)})^{i_{r-1}}$ (2-16)

assume that equation (2-16) yields P independent infinitesimal transformations of the form

$$U_k = \xi_k \partial_x + \eta_k^y \partial_y + \eta_k^{\dot{y}} \partial_{\dot{y}} + \dots + \eta_k^{y^{(r)}} \partial_{y^{(r)}} \quad (2-17)$$

where ξ_k and η_k^y are of the form

$$\begin{aligned} \xi_k &= \sum_{i_1, \dots, i_{r-1}} h_{i_1, \dots, i_{r-1}}(x, y) \dot{y}^{i_1} \ddot{y}^{i_2} \dots (y^{(r-1)})^{i_{r-1}} \\ \eta_k^y &= \sum_{i_1, \dots, i_{r-1}} l_{i_1, \dots, i_{r-1}}(x, y) \dot{y}^{i_1} \ddot{y}^{i_2} \dots (y^{(r-1)})^{i_{r-1}} \end{aligned} \quad (2-18)$$

probably subject to the condition that precisely one function for each non-trivial pair $h_{i_1, i_2, \dots, i_{r-1}}, h_{(i_1-1), i_2, (i_3+1), \dots, (i_{r-1}+(r-2))}$ is specified a priori. This condition would have to be verified to be necessary in each individual case, following the examples given on pages 9 and 12 in the case of second and third order equations.

Following the example of the generalized third order theorem, we set

two points, p_1 and p_2 and a curve α_0 between them to be invariant. Let c_{q_i} represent the highest power of $y^{(r-1)}$ in $\eta_{p_i}^{(q)}$ after all the h.f.z.c. have been eliminated, where $i=1$ or 2 and $q=1, 2, \dots, (r-1)$. Again, following the model of the theorem given above, we set $\sum_{q=1}^{(r-1)} c_{q_1}$ curves with the same $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as α_0 at p_1 and $\sum_{q=r-1}^{2(r-1)} c_{q_2}$ curves with the same $\dot{y}, \ddot{y}, \dots, y^{(r-2)}$ as α_0 at p_2 to be invariant. Using the same logic as was used above we find that the whole plane is now invariant and we are therefore left with $\sum_{q=1}^{2(r-1)} c_{q_1} + m + n + (m+1)^{(r-1)} + (n+1)^{(r-1)} + r$ linearly independent infinitesimal transformations which leave equation (2-16) covariant.

SECTION 3: $\ddot{y}=0$ for $\xi = h_0(x, y) + h_1(x, y) \dot{y} + h_2(x, y) \ddot{y}$

The following is an example of a third order ordinary differential equation which produces the maximum number of generators allowed for its class. There are no h.f.z.c. in this example.

$$\ddot{y}=0 \quad (3-1)$$

$$\xi = h_0(x, y) + h_1(x, y) \dot{y} + h_2(x, y) \ddot{y}$$

$$\eta^4 = h_0(x, y) \quad (3-2)$$

according to the equation on page 12 for this case ($\ddot{y}=0, m=0, n=1, h_3=0$)

$$\begin{aligned} U'''(\ddot{y}) \Big|_{\ddot{y}=0} = & [(h_{0xxxx}) + (3h_{0xyy} - h_{0xxxy})\dot{y} + (3h_{0xy} - 3h_{0xxy} - h_{1xxx})\dot{y}^2 + (h_{0yyy} \\ & - 3h_{0xyy} - 3h_{1xxy})\dot{y}^3 + (-h_{0yyy} - 3h_{1xyy})\dot{y}^4 + (-h_{1yyy})\dot{y}^5] + [(3h_{0xy} - 3h_{0xx}) \\ & + (3h_{0xy} - 9h_{0xy} - 6h_{1xx} - h_{2xxx})\dot{y} + (-6h_{0xy} - 15h_{1xy} - 3h_{2xxy})\dot{y}^2 + (-9h_{1yy} - 3h_{2xyy})\dot{y}^3 \\ & + (-h_{2yyy})\dot{y}^4] \ddot{y} + [(-3h_{0y} - 6h_{1x} - 3h_{2xx}) + (-12h_{1y} - 9h_{2xy})\dot{y} + (-6h_{2yy})\dot{y}^2 + (-3h_{2y})\dot{y}^3] \ddot{y}^2 = 0 \end{aligned}$$

Since \dot{y} and \ddot{y} are arbitrary and independent of each other, we obtain the following set of determining equations

$$l_{0xxx} = 0 \quad (3-1a)$$

$$3l_{0xxy} - l_{0xxx} = 0 \quad (3-1b)$$

$$3l_{0xyy} - 3l_{0xxy} - l_{1xxx} = 0 \quad (3-1c)$$

$$l_{0yyy} - 3l_{0xyy} - 3l_{1xxy} = 0 \quad (3-1d)$$

$$-l_{0yyy} - 3l_{1xyy} = 0 \quad (3-1e)$$

$$-l_{1yyy} = 0 \quad (3-1f)$$

$$3l_{0xy} - 3l_{0xx} = 0 \quad (3-2a)$$

$$3l_{0yy} - 9l_{0xy} - 6l_{1xx} - l_{2xxx} = 0 \quad (3-2b)$$

$$-6l_{0yy} - 15l_{1xy} - 3l_{2xxx} = 0 \quad (3-2c)$$

$$-9l_{1yy} - 3l_{2xyy} = 0 \quad (3-2d)$$

$$-l_{2yyy} = 0 \quad (3-2e)$$

$$-3l_{0y} - 6l_{1x} - 3l_{2xx} = 0 \quad (3-3a)$$

$$-12l_{1y} - 9l_{2xy} = 0 \quad (3-3b)$$

$$-6l_{2yy} = 0 \quad (3-3c)$$

$$-3l_{2y} = 0 \quad (3-4)$$

$$(3-4) \Rightarrow l_2 = \alpha(x) \quad (3-5)$$

$$(3-1a) \Rightarrow l_0 = \beta(y)x^2 + \gamma(y)x + \delta(y) \quad (3-6)$$

$$(3-3b) \Rightarrow l_{1y} = 0 \Rightarrow l_1 = \Theta(x) \quad (3-7)$$

$$(3-2a) \Rightarrow l_{0xx} = 2\beta_y x + \gamma_y \Rightarrow l_0 = \left[\frac{\beta_y}{3}\right]x^3 + \left[\frac{\gamma_y}{2}\right]x^2 + \left[\lambda(y)\right]x + \mu(y) \quad (3-8)$$

$$(3-1d) \Rightarrow (\beta_{yyy} - 3\beta_{yyy})x^2 + (\gamma_{yyy} - 3\gamma_{yyy})x + (\delta_{yyy} - 3\lambda_{yy})$$

Therefore since x is arbitrary

$$-2\beta_{yyy} = 0 \Rightarrow \beta = A_1 y^2 + A_2 y + A_3 \quad (3-9)$$

$$-2\gamma_{yyy} = 0 \Rightarrow \gamma = B_1 y^2 + B_2 y + B_3 \quad (3-10)$$

$$\delta_{yyy} - 3\lambda_{yy} = 0 \Rightarrow \delta_{yyy} = 3\lambda_{yy} \quad (3-11)$$

$$(3-16) \Rightarrow [(12-4)A_1]_y + (6-2)A_2 = 0$$

Therefore since y is arbitrary

$$A_1 = A_2 = 0$$

$$(3-2c) \Rightarrow h_{0yy} = 0 \Rightarrow (\lambda_{yy})_x + \mu_{yy} = 0$$

Thus since x is arbitrary

$$\lambda_{yy} = 0 \Rightarrow \lambda = (E_2)_y + E_3 \quad (3-13)$$

$$\mu_{yy} = 0 \Rightarrow \mu = (F_2)_y + F_3 \quad (3-14)$$

$$(3-3a) \Rightarrow (B_1)_x^2 + (E_2)_x + (2\Theta_x + \alpha_{xx} + F_2) = 0$$

Therefore

$$\alpha_{xx} = (-B_1)_x^2 + (-E_2)_x + (-2\Theta_x - F_2) \quad (3-15)$$

$$(3-1c) \Rightarrow 3(2B_1)_x - 3(2B_1) - \Theta_{xxx} = 0 \Rightarrow \Theta = (C_1)x^2 + (C_2)x + (C_3) \quad (3-16)$$

$$(3-15) \Rightarrow \alpha_{xx} = (-B_1)_x^2 + (-E_2)_x + (-4C_1)_x + (-2C_2 - F_2)$$

Hence

$$\alpha = \left(-\frac{B_1}{12}\right)x^4 + \left(-\frac{E_2}{6} - \frac{2}{3}C_1\right)x^3 + \left(-C_2 - \frac{F_2}{2}\right)x^2 + (D_1)x + (D_2) \quad (3-17)$$

$$(3-2b) \Rightarrow [(6-18+2)B_1]_x + [(-12+4)C_1 + 3\delta_{yy} + (-9+1)E_2] = 0$$

Therefore since x is arbitrary

$$B_1 = 0$$

$$-8C_1 + 3\delta_{yy} - 8E_2 = 0 \Rightarrow \delta_{yy} = \frac{8}{3}C_1 + \frac{8}{3}E_2 \Rightarrow \delta = \left(\frac{4}{3}C_1 + \frac{4}{3}E_2\right)y^2 + G_2 y + G_3 \quad (3-18)$$

Hence we have the result

$$\begin{aligned} U = \eta \partial_x + \gamma \partial_y = & B_2 \left(\frac{x^2}{2} \partial_x + xy \partial_y \right) + E_2 \left(xy \partial_x - \frac{x^3}{6} \partial_x \right. \\ & + \frac{4}{3} y^2 \partial_y \left. \right) + E_3 (x \partial_x) + F_2 (y \partial_x - \frac{x^2}{2} \partial_x) + F_3 (\partial_x) + C_1 (x^2 \partial_x \\ & - \frac{2}{3} x^3 \partial_x + \frac{4}{3} y^2 \partial_y) + C_2 (x \partial_x - x^2 \partial_x) + C_3 (\partial_x) + D_1 (x \partial_x) \\ & + D_2 (\partial_x) + A_3 (x^2 \partial_y) + B_3 (x \partial_y) + G_2 (y \partial_y) + G_3 (\partial_y) \end{aligned}$$

The extended operators are listed in Table I. Note that the first seven operators are those found by Lie²⁵ and the rest are new.

TABLE I

U_1''	$x^2 \partial_y + 2x \partial_{\dot{y}} + 2 \partial_{\ddot{y}}$
U_2''	$\frac{1}{2} x^2 \partial_x + xy \partial_y + y \partial_{\dot{y}} + (\dot{y} - x \ddot{y}) \partial_{\ddot{y}}$
U_3''	$x \partial_y + \partial_{\ddot{y}}$
U_4''	$x \partial_x - \dot{y} \partial_{\dot{y}} - 2 \ddot{y} \partial_{\ddot{y}}$
U_5''	∂_x
U_6''	$y \partial_y + \dot{y} \partial_{\dot{y}} + \ddot{y} \partial_{\ddot{y}}$
U_7''	∂_y
U_8''	$(x^2 \dot{y} - \frac{2}{3} x^3 \ddot{y}) \partial_x + \frac{4}{3} y^2 \partial_y + (\frac{8}{3} y \dot{y} - 2x \dot{y}^2 + x^2 \ddot{y}) \partial_{\dot{y}} + (\frac{2}{3} \dot{y}^2 + \frac{8}{3} y \ddot{y} - 4x \dot{y} \ddot{y} + 2x^2 \ddot{y}^2) \partial_{\ddot{y}}$
U_9''	$(x \dot{y} - x^2 \ddot{y}) \partial_x + (-\dot{y}^2 + x \dot{y} \ddot{y}) \partial_{\dot{y}} + (-2 \dot{y} \ddot{y} + 2x \ddot{y}^2) \partial_{\ddot{y}}$
U_{10}''	$\dot{y} \partial_x - \dot{y} \ddot{y} \partial_{\dot{y}} - 2 \ddot{y}^2 \partial_{\ddot{y}}$
U_{11}''	$x \ddot{y} \partial_x - \dot{y} \ddot{y} \partial_{\dot{y}} - 2 \ddot{y}^2 \partial_{\ddot{y}}$
U_{12}''	$\ddot{y} \partial_x$
U_{13}''	$(x \dot{y} - \frac{x^3}{6} \ddot{y}) \partial_x + \frac{4}{3} y^2 \partial_y + (\frac{5}{3} y \dot{y} - x \dot{y}^2 + \frac{x^2}{2} \ddot{y}) \partial_{\dot{y}} + (\frac{2}{3} \dot{y}^2 + \frac{2}{3} y \ddot{y} - 2x \dot{y} \ddot{y} + x^2 \ddot{y}^2) \partial_{\ddot{y}}$
U_{14}''	$(y - x^2 \ddot{y}) \partial_x + (-\dot{y}^2 + x \dot{y} \ddot{y}) \partial_{\dot{y}} + (-2 \dot{y} \ddot{y} + 2x \ddot{y}^2) \partial_{\ddot{y}}$

Let $X_1 = U_1''$, $X_2 = 2U_2''$, $X_3 = U_3''$, $X_5 = U_5''$, $X_6 = U_6''$, $X_7 = U_7''$, $X_8 = 3U_8''$, $X_9 = U_9''$, $X_{10} = U_{10}''$, $X_{11} = U_{11}''$, $X_{12} = U_{12}''$, $X_{13} = 6U_{13}''$ and $X_{14} = 2U_{14}''$. The commutators for these operators are listed in Table J.*

* David Davison ran these commutators on his computer program Mando (see previous reference).

TABLE J

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}
X_1		0	0	$-2X_1$	$-2X_3$	X_1	0	*	0	*	*	*	*	*
X_2			$-X_1$	$-X_2$	$-2X_4$ $-2X_6$	0	$-2X_3$	*	*	*	*	*	*	*
X_3				$-X_3$	$-X_7$	X_3	0	*	*	*	*	*	*	*
X_4					$-X_5$	0	0	0	$-X_9$	$-2X_{10}$	$-2X_{11}$	$-3X_{12}$	0	$-X_{14}$
X_5						0	0	$6X_9$	X_{10} $-2X_{11}$	0	X_{12}	0	$3X_{14}$	$-2X_{11}$
X_6							$-X_7$	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}
X_7								$8X_6$	0	0	0	0	$16X_6$ $+6X_4$	$2X_5$
X_8									*	*	*	*	*	*
X_9										*	*	*	*	*
X_{10}											0	*	*	0
X_{11}												*	*	*
X_{12}													*	*
X_{13}														*
X_{14}														

The asterisks in Table J stand for generators of a form not allowed in the original set. See Appendix C for a list of the asterisked commutators.

The set of generators of $\ddot{y}=0$ does not close under a finite number of commutations. This fact is shown below.

Let $X_n^{''}$ be any generator of the form

$$X_n^{''} = A \ddot{y}^n \partial x \quad \text{where } A = \text{const.} \neq 0, n > 0$$

Take the commutator of U_{11}'' with X_n''

$$[U_{11}'', X_n''] = [x\ddot{y}\partial_x - \dot{y}\ddot{y}\partial_{\dot{y}} - 2\ddot{y}^2\partial_{\ddot{y}}, A\ddot{y}^m\partial_x] = -A\ddot{y}^{(m+1)}\partial_x - 2Am\ddot{y}^{(m+1)}\partial_x \\ = -(2m+1)A\ddot{y}^{(m+1)}\partial_x$$

The commutator is of the same form as, but with a greater power of \ddot{y}

than, X_n'' . Therefore $[U_{11}'', X_n'']$, $[U_{11}'', [U_{11}'', X_n'']]$, and so on, produce higher and higher powers of \ddot{y} and therefore the set $\{U_{11}'', X_n''\}$

does not close under a finite number of commutations. Now $X_1'' = U_{12}''$

(where $A=1$). Hence the generators listed in Table I do not close under a finite number of commutations.

This proof also applies to the example given in Section 4 of this chapter, since the set of generators for that example includes all of the generators given in Table I.

SECTION 4: $\ddot{y}=0$ for $\xi = h_0(x,y) + h_1(x,y)\dot{y} + h_2(x,y)\ddot{y} + h_3(x,y)\ddot{\dot{y}}$

$$\ddot{y}=0$$

$$\xi = h_0(x,y) + h_1(x,y)\dot{y} + h_2(x,y)\ddot{y} + h_3(x,y)\ddot{\dot{y}} \\ \eta^y = l_0(x,y)$$

From the equation given on page 12 for this case ($f=0$, $m=0$, $n=1$) we

obtain the following equation

$$U'''(\ddot{y}) \Big|_{\ddot{y}=0} = [(l_{0xxx}) + (-h_{0xxx} + 3l_{0xx\dot{y}})\dot{y} + (-h_{1xxx} - 3h_{0xx\dot{y}} + 3l_{0x\dot{y}\dot{y}})\dot{y}^2 \\ + (-3h_{1xx\dot{y}} - 3h_{0x\dot{y}\dot{y}} + l_{0y\dot{y}\dot{y}})\dot{y}^3 + (-3h_{1x\dot{y}\dot{y}} - h_{0y\dot{y}\dot{y}})\dot{y}^4 + (-h_{1y\dot{y}\dot{y}})\dot{y}^5] \\ + [(-3h_{0xxx} + 3l_{0x\dot{y}}) + (-h_{2xxx} - 3h_{2x\dot{y}\dot{y}} - 9h_{1y\dot{y}})\dot{y}^3 + (-3h_{3x\dot{y}\dot{y}} - h_{2y\dot{y}\dot{y}})\dot{y}^4 \\ + (-h_{3y\dot{y}\dot{y}})\dot{y}^5] \ddot{y} + [(-3h_{2xxx} - 6h_{1xx} - 3h_{0y\dot{y}}) + (-6h_{3xx} - 9h_{2x\dot{y}} - 12h_{1y})\dot{y} \\ + (-15h_{3x\dot{y}} - 6h_{2y\dot{y}})\dot{y}^2 + (-9h_{3y\dot{y}})\dot{y}^3] \ddot{\dot{y}}^2 + [(-6h_{3xx} - 3h_{2y}) \\ + (-12h_{3y})\dot{y}] \ddot{\dot{y}}^3 = 0$$

Therefore, since \dot{y} and \ddot{y} are arbitrary and algebraically independent we obtain the following set of determining equations

$$l_{0xxx} = 0 \quad (4-1)$$

$$-l_{0xxx} + 3l_{0xxxy} = 0 \quad (4-2)$$

$$-l_{1xxx} - 3l_{0xxxy} + 3l_{0xyyy} = 0 \quad (4-3)$$

$$-3l_{1xxy} - 3l_{0xyyy} + l_{0yyyy} = 0 \quad (4-4)$$

$$-3l_{1xyy} - l_{0yyyy} = 0 \quad (4-5)$$

$$-l_{1yyyy} = 0 \quad (4-6)$$

$$-3l_{0xx} + 3l_{0xy} = 0 \quad (4-7)$$

$$-l_{2xxx} - 6l_{1xx} - 9l_{0xy} + 3l_{0yy} = 0 \quad (4-8)$$

$$-l_{3xxx} - 3l_{2xxy} - 15l_{1xy} - 6l_{0yy} = 0 \quad (4-9)$$

$$-3l_{3xxy} - 3l_{2xyy} - 9l_{1yy} = 0 \quad (4-10)$$

$$-3l_{3xyy} - l_{2yyy} = 0 \quad (4-11)$$

$$-l_{3yyy} = 0 \quad (4-12)$$

$$-3l_{2xx} - 6l_{1x} - 3l_{0y} = 0 \quad (4-13)$$

$$-6l_{3xx} - 9l_{2xy} - 12l_{1y} = 0 \quad (4-14)$$

$$-15l_{3xy} - 6l_{2yy} = 0 \quad (4-15)$$

$$-9l_{3yy} = 0 \quad (4-16)$$

$$-6l_{3x} - 2l_{2y} = 0 \quad (4-17)$$

$$-12l_{3y} = 0 \quad (4-18)$$

$$(4-18) \Rightarrow l_{3y} = 0 \Rightarrow l_3 = \alpha(x) \quad (4-19)$$

$$(4-6) \Rightarrow l_{1yyyy} = 0 \Rightarrow l_1 = [\beta(x)]y^2 + [\gamma(x)]y + [\delta(x)] \quad (4-20)$$

$$(4-1) \Rightarrow l_{0xxx} = 0 \Rightarrow l_0 = [\lambda(y)]x^2 + [\epsilon(y)]x + [\theta(y)] \quad (4-21)$$

$$(4-2) \Rightarrow l_{0xxx} = 3(2\lambda_y) \Rightarrow l_0 = [\lambda_y]x^3 + [\mu(y)]x^2 + [\nu(y)]x + [\pi(y)] \quad (4-22)$$

$$(4-17) \Rightarrow l_{2y} = -2(\alpha_x) \Rightarrow l_2 = [-2\alpha_x]y + [\zeta(x)] \quad (4-23)$$

$$(4-5) \Rightarrow 3(2\beta_x) + [\lambda_{yyy}]x^3 + [\mu_{yyy}]x^2 + [\nu_{yyy}]x + [\pi_{yyy}] = 0$$

Therefore since β is a function of x only

$$\lambda_{yyy} = \text{const.} \Rightarrow \lambda = (D_1)y^4 + (D_2)y^3 + (D_3)y^2 + (D_4)y + (D_5) \quad (4-24)$$

$$\mu_{yyy} = \text{const.} \Rightarrow \mu = (A_1)y^3 + (A_2)y^2 + (A_3)y + (A_4) \quad (4-25)$$

$$\nu_{yyy} = \text{const.} \Rightarrow \nu = (B_1)y^3 + (B_2)y^2 + (B_3)y + (B_4) \quad (4-26)$$

$$\pi_{yyy} = \text{const.} \Rightarrow \pi = (C_1)y^3 + (C_2)y^2 + (C_3)y + (C_4) \quad (4-27)$$

$$(4-7) \Rightarrow [(-6+2)\lambda_y]x + [-2\mu + \epsilon_y] = 0$$

Hence since x is arbitrary

$$\lambda_y = 0 \Rightarrow \lambda = D_5 \Rightarrow D_1 = D_2 = D_3 = D_4 = 0 \quad (4-28)$$

$$-2\mu + \epsilon_y = 0 \Rightarrow \epsilon_y = 2\mu \Rightarrow \epsilon = \left(\frac{1}{2}A_1\right)y^4 + \left(\frac{2}{3}A_2\right)y^3 + (A_3)y^2 + (2A_4)y + F \quad (4-29)$$

$$(4-14) \Rightarrow [8\beta(x)]y + [-\alpha_{xx} + 4\gamma(x)] = 0$$

Thus since y is arbitrary

$$\beta = 0$$

$$-\alpha_{xx} + \gamma(x) = 0 \Rightarrow \alpha_{xx} = \gamma(x) \quad (4-30)$$

$$(4-13) \Rightarrow [\epsilon_{xxx} + 2\delta_x] + [\lambda_{yy}]x^3 + [\mu_y]x^2 + [\nu_y]x + [\pi_y] = 0$$

Therefore since ϵ and δ are functions of x only

$$\lambda_{yy} = \text{const.}$$

$$\mu_y = \text{const.} \Rightarrow \mu = (A_3)y + (A_4) \Rightarrow A_1 = A_2 = 0 \quad (4-31)$$

$$\nu_y = \text{const.} \Rightarrow \nu = (B_3)y + (B_4) \Rightarrow B_1 = B_2 = 0 \quad (4-32)$$

$$\pi_y = \text{const.} \Rightarrow \pi = (C_3)y + (C_4) \Rightarrow C_1 = C_2 = 0 \quad (4-33)$$

Hence

$$\epsilon_{xxx} + 2\delta_x = (-A_3)x^2 + (-B_3)x + (-C_3) \quad (4-34)$$

$$(4-4) \Rightarrow (-3\gamma_{xx}) + (\epsilon_{yyy})x + (\Theta_{yyy}) = 0$$

Thus since γ depends on x only

$$\epsilon_{yyy} = \text{const.}$$

$$\Theta_{yyy} = \text{const.} \Rightarrow \Theta = (Z_1)y^3 + (Z_2)y^2 + (Z_3)y + (Z_4) \quad (4-35)$$

Therefore

$$\gamma_{xx} = 2Z_1 \Rightarrow \gamma = (Z_1)x^2 + (E_1)x + E_2 \quad (4-36)$$

$$(-\delta_{xxx}) + (-6A_3) + (6A_3) = 0$$

Thus

$$\delta_{xxx} = 0 \Rightarrow \delta = (G_1)x^2 + (G_2)x + (G_3) \quad (4-37)$$

$$(4-37) \Rightarrow \delta_{xxx} + (4G_1)x + (2G_2) = (-A_3)x^2 + (-B_3)x + (-C_3)$$

Hence

$$\delta = (-\frac{A_3}{12})x^4 + (-\frac{B_3}{6} - \frac{2}{3}G_1)x^3 + (-\frac{C_3}{2} - G_2)x^2 + (H_1)x + (H_2) \quad (4-38)$$

$$(4-8) \Rightarrow [(4-12+18)Z_1]y + [(2-18+6)A_3]x + [(1-9)B_3 + (4-12)G_1 + 6Z_2] = 0$$

Therefore since x and y are independent

$$Z_1 = 0, A_3 = 0$$

$$-8B_3 - 8G_1 + 6Z_2 = 0 \Rightarrow Z_2 = \frac{4}{3}B_3 + \frac{4}{3}G_1 \quad (4-39)$$

$$(4-9) \Rightarrow (1-6+15)\frac{1}{3} = 0 \Rightarrow \gamma = E_2 \Rightarrow E_1 = 0 \quad (4-40)$$

$$(4-30) \Rightarrow \alpha_{xx} = E_2 \Rightarrow \alpha = (\frac{E_2}{2})x^2 + (I_1)x + (I_2) \quad (4-41)$$

Hence we have the result

$$\begin{aligned} U = & A_4(x^2\partial_x + 2xy\partial_y) + B_4(x\partial_x) + C_3(y\partial_x - \frac{x^2}{2}\partial_y) + C_4(\partial_x) \\ & + G_1(x^2y\partial_x - \frac{2}{3}x^3\partial_y + \frac{4}{3}y^2\partial_y) + G_2(xy\partial_x - x^2\partial_y) + E_2(y\partial_x - 2xy\partial_y + \frac{x^2}{2}\partial_y) \\ & + G_3(y\partial_x) + H_1(xy\partial_x) + I_1(-2yy\partial_x + xy\partial_y) + H_2(y\partial_x) + I_2(y\partial_y) + D_5(x^2\partial_y) \\ & + F(x\partial_y) + Z_3(y\partial_y) + Z_4(\partial_y) + B_3(xy\partial_x - \frac{x^3}{6}\partial_y + \frac{4}{3}y^2\partial_y) \end{aligned}$$

Thus we recover all of the generators listed in Table I plus three new ones which are listed in Table K.

TABLE K

U_{15}''	$(y\partial_y - 2xy\partial_x + \frac{x^2}{2}\partial_y)\partial_x + (-y^3 + y\partial_y^2 + xy^2\partial_y - \frac{x^2}{2}\partial_y^2)\partial_y + (-2y^2\partial_y + 2y\partial_y^2 + 2xy\partial_y^2 - x^2\partial_y^3)\partial_y$
U_{16}''	$(-2y\partial_y + x\partial_y^2)\partial_x + (y^2\partial_y - xy\partial_y^2)\partial_y + (2y\partial_y^2 - 2xy\partial_y^3)\partial_y$
U_{17}''	$y\partial_y^2\partial_x - y\partial_y^3\partial_y - 2y^3\partial_y^2$

Note that in this case there are h.f.z.c. in both $\eta^{\dot{y}}$ and $\eta^{\ddot{y}}$.

The generators for this case do not close under a finite number of commutations. The proof of this was given on page 57 and applies here since the generators listed in Table I are included in the set of generators for this case.

CHAPTER V: Conclusion

These four theorems (including Lie's two) have much in common. All are based on the premise that if one fixes enough points and curves the entire plane will become invariant under the action of a linear combination of operators which also leaves a differential equation covariant. All make use of the property of the Vandermonde determinant that it cannot vanish if all of its components are different. All make use of the maximum number of derivatives of y needed to specify a curve. All use the particular form of the $\eta^{y^{(q)}}$ up to $q=r-1$ (where r is the order of the original differential equation) in terms of powers of $y^{(r-1)}$.

However there are also important differences, some of which link up certain pairs of the theorems as opposed to other pairs. For example, the theorems in Chapter I are proofs by contradiction, whereas the theorems of Chapter II are direct proofs. Another way of pairing the theorems is Lie versus generalized. The generalized theorems differ from Lie's in several essential ways. In the first place, Lie allowed no $y^{(q)}$ in ξ or η and therefore it always took two and only two equations to fix a point; whereas in the AKW case the number of equations needed to fix a point depends upon the number of different derivatives that exist in ξ and η and on the powers of those $y^{(q)}$'s. Moreover whether or not a $y^{(q)}$ is considered to be fixed or not affects the number of equations needed to fix a point. A second way in which Lie differs from the generalized version is that the Vandermonde determinant

is of a higher order in the AKW case. This is especially noticeable in Lie's r^{th} order ($r > 2$) counting theorem, where none of his ($q=1,2,\dots,(r-2)$) depend on $y^{(r-1)}$ and $\eta^{y^{(r-1)}}$ is linear in $y^{(r-1)}$ for all possible r 's. This means that his theorem is greatly simplified since he needs to use a determinant only once and even then it is so obviously not zero --as it is merely of order two-- that in his original paper Lie never mentions determinants but rather simply states that since he has two equations that are linear in $y^{(r-1)}$ (where two distinct values of $y^{(r-1)}$ are used) the coefficients of $y^{(r-1)}$ must be zero.

In its use of Vandermonde determinants the generalized third order counting theorem is more like Lie's second order theorem than his r^{th} order ($r > 2$) theorem.

Lie completely avoids consideration of f.z.c. He can do this because his ξ and η^y never appear in the $\eta^{y^{(q)}}$ except when differentiated. In the AKW η^y , on the other hand, the h_i^y 's and λ_i^y 's are not always differentiated, so that setting ξ and η^y to zero might produce f.z.c. The reason h and λ are in some cases not differentiated is best illustrated by example.

Consider

$$\eta^y = \lambda_1(x, y) \dot{y}$$

Therefore

$$\frac{d\eta^y}{dx} = \lambda_{1x} \dot{y} + \lambda_{1y} \dot{y}^2 + \lambda_1 \ddot{y}$$

As is shown in the above equation, the derivative of the product $\lambda_1 \dot{y}$ produces a term in which λ_1 is not differentiated. No such products appear in ξ or η^y for the Lie case and therefore in all $\eta^{y^{(q)}}$ ($q \geq 1$) --which are derivatives of η^y plus a term $(-y^{(q)} \xi_1)$ which cannot produce

undifferentiated terms--- ξ and η^y must be differentiated at least once.

Another important difference between the Lie theorems and the generalized ones is that f , i.e., the value of $y^{(r)}$, never appears in the Lie theorems. This occurs because f only appears in $\eta^{y^{(r)}}$ for Lie, but in the theorems he only uses $\eta^{y^{(q)}}$ up to and including $q=r-1$. Even in the generalized theorem for second order equations f appears in η^y , although it has a coefficient of zero in the theorem and therefore the f disappears. However, in the sense that the generalized second order counting does not depend on the powers of \dot{y} in f , that theorem has more in common with Lie's second order theorem than with the generalized theorem for third order equations.

When Lie wrote his theorems there were two important differences between them:

- a) the theorem for second order contained an η^y which was not linear in \dot{y} , while in the other case $\eta^{y^{(r-1)}}$ was linear in $y^{(r-1)}$.
- b) in the second order theorem only points were set invariant, whereas in the r^{th} order ($r > 2$) theorem both points and curves were fixed.

In the generalized version only difference (b) remains.

Note that in the AKW examples used in this work there is a certain lopsidedness to the form allowed for the generators, as in all cases η^y depends only on x and y while ξ depends on $x, y, \dot{y}, \dots, y^{(r-1)}$. Some kind of unsymmetrical arrangement ---although not necessarily this one--- must be used because otherwise we obtain pairs of arbitrary functions, as was demonstrated in Chapter II. Symmetry can be restored, however, through the use of commutators. The commutators listed in the appendices demonstrate by example that some of the generators which appear most

in commutation are those which restore the symmetry of the transformations --i.e., those in which η^y also depends on $x, y, \dot{y}, \dots, y^{(r-1)}$.

The utility of a counting theorem depends upon the utility of the generators --as such a theorem is used mainly as a checking device when one is trying to find all the independent generators which leave a particular equation covariant. The infinitesimal transformations can be used to generate finite transformations or to generate solutions to differential equations --either directly by integration or by acting on known solutions.

The particular form chosen for ξ_i and η^y in these generalizations is probably the most natural for the theorems since the proofs depend on the use of Vandermonde determinants which have to do with power series.

The next logical extension would be to use functions that can be expanded as infinite power series in the derivatives of y --e.g. $\sin \dot{y}$.²⁶ At the moment this presents difficulties, however, since it is difficult to deal with a power series which is infinite. In some cases, however, a truncated series, that is, an approximation to the infinite series might be useful. These theorems do not exhaust all possibilities for counting theorems even for the given form of ξ_i and η^y . For example, there exist no counting theorems for systems of equations in more than one dimension for the AKW extension, despite fact that this case is of great importance in physics. This is an area which is presently being investigated.

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17. Ibid., 298. See also Cohen, 146.
18. Cohen, 207.
19. Taken from Anderson and Davison.
20. Ovsjannikov
21. Lie, Vorlesungen Über Continuerliche Gruppen, 298. See also
Cohen, 146.
22. Ibid., 296.
23. Ibid., 115.
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APPENDIX A: Asterisked Commutators From Table D

$$[U''_{15}, U''_{14}] = \left(\frac{3}{2} x^2 y^3 - 3 x^3 y^2 \dot{y} + 2 x^4 y \dot{y}^2 - \frac{x^5 \dot{y}^3}{2} \right) \partial_x + \left(\frac{x y^4}{2} - \frac{x^2 y^3 \dot{y}}{2} \right) \partial_y + \left(3 x^2 y^2 \dot{y}^2 - 2 x^3 y \dot{y}^3 + \frac{x^4 \dot{y}^4}{2} + \frac{y^4}{2} \right) \partial_{\dot{y}}$$

$$[U''_{15}, U''_{13}] = \left(\frac{3}{2} x y^3 - 3 x^2 y^2 \dot{y} + 2 x^3 y \dot{y}^2 - \frac{x^4 \dot{y}^3}{2} \right) \partial_x + \left(\frac{y^4}{2} - \frac{x y^3 \dot{y}}{2} \right) \partial_y$$

$$[U''_{15}, U''_{12}] = \left(-2 x^2 y \dot{y} + x^3 \dot{y}^2 + \frac{x y^2}{2} \right) \partial_x + \left(-\frac{x y^2 \dot{y}}{2} \right) \partial_y + \left(-y^2 \dot{y} + 2 x y \dot{y}^2 - x^2 \dot{y}^3 \right) \partial_{\dot{y}}$$

$$[U''_{15}, U''_{11}] = \left(-\frac{5}{2} x y^2 \dot{y} + 2 x^2 y \dot{y}^2 - \frac{x^3 \dot{y}^3}{2} + \frac{y^3}{2} \right) \partial_x + \left(-\frac{y^3 \dot{y}}{2} \right) \partial_y + \left(-\frac{y^2 \dot{y}^2}{2} + x y \dot{y}^3 - \frac{x^2 \dot{y}^4}{2} \right) \partial_{\dot{y}}$$

$$[U''_{15}, U''_{10}] = \left(-3 x y \dot{y} + 2 x^2 \dot{y}^2 + \frac{y^2}{2} \right) \partial_x + \left(-\frac{y^2 \dot{y}}{2} \right) \partial_y + \left(y \dot{y}^2 - x \dot{y}^3 \right) \partial_{\dot{y}}$$

$$[U''_{15}, U''_9] = \left(-\frac{x^2 y^2}{2} \right) \partial_x + \left(-\frac{x^2 y^2 \dot{y}}{2} \right) \partial_y$$

$$[U''_{15}, U''_8] = \left(-x^2 y^2 + \frac{3}{2} x^3 y \dot{y} - \frac{x^4 \dot{y}^2}{2} \right) \partial_x + \left(-\frac{x y^3}{2} + \frac{x^2 y^2 \dot{y}}{2} \right) \partial_y + \left(-\frac{3}{2} x^2 y \dot{y}^2 + \frac{x^3 \dot{y}^3}{2} + \frac{3}{2} x y^2 \dot{y} - \frac{y^3}{2} \right) \partial_{\dot{y}}$$

$$[U''_{15}, U''_7] = \left(-\frac{x^3 y}{2} \right) \partial_x + \left(-\frac{x^3 y \dot{y}}{2} \right) \partial_y$$

$$[U''_{15}, U''_6] = \left(-\frac{x^3}{2} \right) \partial_x + \left(-\frac{x^3 \dot{y}}{2} \right) \partial_y$$

$$[U''_{15}, U''_5] = \left(-\frac{3}{2} x y^2 + \frac{3}{2} x^2 y \dot{y} - \frac{x^3 \dot{y}^2}{2} \right) \partial_x + \left(-\frac{y^3}{2} \right) \partial_y$$

$$[U''_{14}, U''_{12}] = \left(2 x^2 y \dot{y}^2 - x y^2 \dot{y} - x^3 \dot{y}^3 \right) \partial_x + \left(y^2 \dot{y}^2 - 2 x y \dot{y}^3 + x^2 \dot{y}^4 \right) \partial_{\dot{y}}$$

$$[U''_{14}, U''_{10}] = \left(y^2 \dot{y} - x^2 \dot{y}^3 \right) \partial_x + \left(-2 y \dot{y}^3 + 2 x \dot{y}^4 \right) \partial_{\dot{y}}$$

$$[U''_{14}, U''_9] = \left(-6 x^2 y^2 \dot{y} + 4 x y^3 + 4 x^3 y \dot{y}^2 - x^4 \dot{y}^3 \right) \partial_x + \left(y^4 \right) \partial_y$$

$$[U''_{14}, U''_8] = \left(3 x^2 y^2 \dot{y} - 3 x^3 y \dot{y}^2 - x y^3 + x^4 \dot{y}^3 \right) \partial_x + \left(-3 x y^2 \dot{y}^2 + 3 x^2 y \dot{y}^3 + y^3 \dot{y} - x^3 \dot{y}^4 \right) \partial_{\dot{y}}$$

$$[U''_{14}, U''_7] = \left(x^2 y^2 \right) \partial_x + \left(x^2 y^2 \dot{y} \right) \partial_y$$

$$[U''_{14}, U''_6] = \left(3 x^2 y - x^3 \dot{y} \right) \partial_x + \left(x y^2 + x^2 y \dot{y} \right) \partial_y + \left(-2 x y \dot{y} + y^2 + x^2 \dot{y}^2 \right) \partial_{\dot{y}}$$

$$[U''_{14}, U''_5] = \left(y^3 - 2 x y^2 \dot{y} + x^2 y \dot{y}^2 \right) \partial_x + \left(-y^2 \dot{y}^2 + 2 x y \dot{y}^3 - x^2 \dot{y}^4 \right) \partial_{\dot{y}}$$

$$[U''_{13}, U''_{12}] = \left(-y^2 \dot{y} + 2 x y \dot{y}^2 - x^2 \dot{y}^3 \right) \partial_x$$

$$[u''_{13}, u''_{10}] = (-x\dot{y}^3) \partial_x + (\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{13}, u''_9] = (-5xy^2\dot{y} + 2y^3 + 4x^2y\dot{y}^2 - x^3\dot{y}^3) \partial_x + (-y^2\dot{y}^2 + 2xy\dot{y}^3 - x^2\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{13}, u''_8] = (-y^3 + 3xy^2\dot{y} - 3x^2y\dot{y}^2 + x^3\dot{y}^3) \partial_x$$

$$[u''_{13}, u''_7] = (-xy^2 + x^2y\dot{y}) \partial_x + (-y^3 + xy^2\dot{y}) \partial_y + (-y^2\dot{y} + 2xy\dot{y}^2 - x^2\dot{y}^3) \partial_{\dot{y}}$$

$$[u''_{13}, u''_3] = (xy - x^2\dot{y}) \partial_x + (-y^2 + xy\dot{y}) \partial_y + (-2y\dot{y} + 2x\dot{y}^2) \partial_{\dot{y}}$$

$$[u''_{13}, u''_1] = (-y^2\dot{y} + xy\dot{y}^2) \partial_x + (y\dot{y}^3 - x\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{12}, u''_{11}] = (-2y\dot{y}^2 + x\dot{y}^3) \partial_x + (\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{12}, u''_{10}] = (-2\dot{y}^2) \partial_x$$

$$[u''_{12}, u''_9] = (2x^2\dot{y}^2 - 2xy\dot{y}) \partial_x + (-2x\dot{y}^3 + 2y\dot{y}^2) \partial_{\dot{y}}$$

$$[u''_{12}, u''_8] = (-x^2\dot{y}^2 + xy\dot{y}) \partial_x + (x\dot{y}^3 - y\dot{y}^2) \partial_{\dot{y}}$$

$$[u''_{12}, u''_7] = (2x^2\dot{y} - xy) \partial_x + (xy\dot{y}) \partial_y + (2y\dot{y} - 2x\dot{y}^2) \partial_{\dot{y}}$$

$$[u''_{12}, u''_3] = (-x) \partial_x + (x\dot{y}) \partial_y + (2\dot{y}) \partial_{\dot{y}}$$

$$[u''_{12}, u''_1] = (-y\dot{y} + x\dot{y}^2) \partial_x$$

$$[u''_{11}, u''_{10}] = (-\dot{y}^3) \partial_x$$

$$[u''_{11}, u''_9] = (4xy\dot{y}^2 - 3y^2\dot{y} - x^2\dot{y}^3) \partial_x + (2y\dot{y}^3 - 2x\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{11}, u''_8] = (2y^2\dot{y} - 3xy\dot{y}^2 + x^2\dot{y}^3) \partial_x + (-y\dot{y}^3 + x\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{11}, u''_7] = (2xy\dot{y} - y^2) \partial_x + (y^2\dot{y}) \partial_y + (2y\dot{y}^2 - 2x\dot{y}^3) \partial_{\dot{y}}$$

$$[u''_{11}, u''_3] = (-x\dot{y} - y) \partial_x + (y\dot{y}) \partial_y + (3\dot{y}^2) \partial_{\dot{y}}$$

$$[u''_{11}, u''_1] = (y\dot{y}^2) \partial_x + (-\dot{y}^4) \partial_{\dot{y}}$$

$$[u''_{10}, u''_9] = (-2y\dot{y} + 4x\dot{y}^2) \partial_x + (-2\dot{y}^3) \partial_{\dot{y}}$$

$$[u''_{10}, u''_8] = (2y\dot{y} - 3x\dot{y}^2) \partial_x + (\dot{y}^3) \partial_{\dot{y}}$$

$$[u''_{10}, u''_7] = (3x\dot{y} - y) \partial_x + (y\dot{y}) \partial_y + (-\dot{y}^2) \partial_{\dot{y}}$$

$$[u''_{10}, u''_3] = (-1) \partial_x + (\dot{y}) \partial_y$$

$$[u''_{10}, u''_1] = (\dot{y}^2) \partial_x$$

$$[u''_9, u''_8] = (-2x^2y\dot{y} + xy^2 + x^3\dot{y}^2)\partial_x + (2xy\dot{y}^2 - x^2\dot{y}^3 - y^2\dot{y})\partial_{\dot{y}}$$

$$[u''_9, u''_7] = (-x^2y + x^3\dot{y})\partial_x + (x^2y\dot{y} - xy^2)\partial_y + (-x^2\dot{y}^2 - y^2 + 2xy\dot{y})\partial_{\dot{y}}$$

$$[u''_9, u''_3] = (-x^2)\partial_x + (x^2\dot{y} - 2xy)\partial_y + (-2y + 2x\dot{y})\partial_{\dot{y}}$$

$$[u''_9, u''_1] = (-2xy\dot{y} + x^2\dot{y}^2 + y^2)\partial_x$$

$$[u''_8, u''_7] = (x^2y - x^3\dot{y})\partial_x + (xy^2 - x^2y\dot{y})\partial_y + (-2xy\dot{y} + x^2\dot{y}^2 + y^2)\partial_{\dot{y}}$$

$$[u''_8, u''_3] = (xy - x^2\dot{y})\partial_y + (y - x\dot{y})\partial_{\dot{y}}$$

$$[u''_8, u''_1] = (-y^2 + 2xy\dot{y} - x^2\dot{y}^2)\partial_x$$

APPENDIX B: Asterisked Commutators From Table G

In this Appendix the following notation is used:

$$\begin{aligned} S &= \sin(\sqrt{k}x), \quad C = \cos(\sqrt{k}x) \\ \bar{S} &= \sin(2\sqrt{k}x), \quad \bar{C} = \cos(2\sqrt{k}x) \\ \bar{\bar{S}} &= \sin(3\sqrt{k}x), \quad \bar{\bar{C}} = \cos(3\sqrt{k}x) \\ K &= \sqrt{k} \end{aligned}$$

$$\begin{aligned} [u_1'', u_9''] &= (4Ky S\bar{S} + yS\bar{C} + Ky C\bar{C})\partial_x + (-2K^2y^2\bar{S}C - Ky y C\bar{C} \\ &\quad + 3K^2y^2S\bar{C})\partial_y + (-4K^3y^2S\bar{S} + K^2y y S\bar{C} - 3Ky^2C\bar{C} - 2Ky^2S\bar{S} \\ &\quad - 4K^2y y S\bar{C})\partial_{\dot{y}} + (2K^4y^2\bar{S}C + K^3y y C\bar{C} - 3K^4y^2S\bar{C})\partial_{\ddot{y}} \end{aligned}$$

$$\begin{aligned} [u_1'', u_{10}''] &= (-4Ky S\bar{C} + yS\bar{S} + Ky \bar{S}C)\partial_x + (2K^2y^2C\bar{C} - Ky y \bar{S}C + 3K^2y^2S\bar{S})\partial_y \\ &\quad + (4K^3y^2S\bar{C} + K^2y y S\bar{S} - 3Ky^2\bar{S}C + 2Ky^2S\bar{C} + 4K^2y y C\bar{C})\partial_{\dot{y}} \\ &\quad + (-2K^4y^2C\bar{C} + K^3y y \bar{S}C - 3K^4y^2S\bar{S})\partial_{\ddot{y}} \end{aligned}$$

$$\begin{aligned} [u_1'', u_{11}''] &= (K\bar{C}^2)\partial_x + (-Ky C^2 - 2K^2y S\bar{C})\partial_y + (2K^2y S\bar{C} - K^3y C^2 + 2K^3y S^2)\partial_{\dot{y}} \\ &\quad + (K^3y C^2 + 2K^4y S\bar{C})\partial_{\ddot{y}} \end{aligned}$$

$$\begin{aligned} [u_1'', u_{12}''] &= (KSC)\partial_x + (-Ky SC - 2K^2y S^2)\partial_y + (-3Ky SC - 2K^2y C^2)\partial_{\dot{y}} \\ &\quad + (K^3y SC + 2K^4y S^2)\partial_{\ddot{y}} \end{aligned}$$

$$\begin{aligned} [u_1'', u_{13}''] &= (2KS\bar{S} + KC\bar{C})\partial_x + (-2K^2y \bar{S}C - Ky C\bar{C} + 2K^2y S\bar{C})\partial_y \\ &\quad + (-4K^3y S\bar{S} - 2K^2y S\bar{C} + 2K^2y \bar{S}C - 3K^3y C\bar{C})\partial_{\dot{y}} + (2K^4y \bar{S}C \\ &\quad + K^3y C\bar{C} - 2K^4y S\bar{C})\partial_{\ddot{y}} \end{aligned}$$

$$\begin{aligned} [u_1'', u_{14}''] &= (-2KS\bar{C} + K\bar{S}C)\partial_x + (2K^2y C\bar{C} - Ky \bar{S}C + 2K^2y S\bar{S})\partial_y + (4K^3y S\bar{C} \\ &\quad - 2K^2y S\bar{S} - 2K^2y C\bar{C} - 3K^3y \bar{S}C)\partial_{\dot{y}} + (-2K^4y C\bar{C} + K^3y \bar{S}C \\ &\quad - 2K^4y S\bar{S})\partial_{\ddot{y}} \end{aligned}$$

$$[u_1'', u_{15}''] = (\dot{y}S + KyC)\partial_x + (Ky\dot{y}C - 3K^2y^2S)\partial_y + (-3K^2y\dot{y}S - 3K\dot{y}^2C - 2K^3y^2C)\partial_{\dot{y}} + (K^3y\dot{y}C + 3K^4y^2S)\partial_{\ddot{y}}$$

$$[u_2'', u_9''] = (4Ky\bar{S}C + \dot{y}C\bar{C} - KyS\bar{C})\partial_x + (2K^2y^2S\bar{S} + Ky\dot{y}S\bar{C} + 3K^2y^2C\bar{C})\partial_y + (-4K^3y^2\bar{S}C + K^2y\dot{y}C\bar{C} + 3K\dot{y}^2S\bar{C} - 2K\dot{y}^2\bar{S}C + 4K^2y\dot{y}S\bar{S})\partial_{\dot{y}} + (-2K^4y^2S\bar{S} - K^3y\dot{y}S\bar{C} - 3K^4y^2C\bar{C})\partial_{\ddot{y}}$$

$$[u_2'', u_{10}''] = (-4KyC\bar{C} + \dot{y}\bar{S}C - KyS\bar{S})\partial_x + (-2K^2y^2S\bar{C} + Ky\dot{y}S\bar{S} + 3K^2y^2\bar{S}C)\partial_y + (4K^3y^2C\bar{C} + K^2y\dot{y}\bar{S}C + 3K\dot{y}^2S\bar{S} + 2K\dot{y}^2C\bar{C} - 4K^2y\dot{y}S\bar{C})\partial_{\dot{y}} + (2K^4y^2S\bar{C} - K^3y\dot{y}S\bar{S} - 3K^4y^2\bar{S}C)\partial_{\ddot{y}}$$

$$[u_2'', u_{11}''] = (-KSC)\partial_x + (KySC - 2K^2yC^2)\partial_y + (3K^3ySC - 2K^2\dot{y}S^2)\partial_{\dot{y}} + (-K^3\dot{y}SC + 2K^4yC^2)\partial_{\ddot{y}}$$

$$[u_2'', u_{12}''] = (-KS^2)\partial_x + (KyS^2 - 2K^2ySC)\partial_y + (2K^2\dot{y}SC + K^3yS^2 - 2K^3yC^2)\partial_{\dot{y}} + (-K^3\dot{y}S^2 + 2K^4ySC)\partial_{\ddot{y}}$$

$$[u_2'', u_{13}''] = (2K\bar{S}C - KS\bar{C})\partial_x + (2K^2yS\bar{S} + KyS\bar{C} + 2K^2yC\bar{C})\partial_y + (-4K^3y\bar{S}C - 2K^2\dot{y}C\bar{C} - 2K^2\dot{y}S\bar{S} + 3K^3yS\bar{C})\partial_{\dot{y}} + (-2K^4yS\bar{S} - K^3\dot{y}S\bar{C} - 2K^4C\bar{C})\partial_{\ddot{y}}$$

$$[u_2'', u_{14}''] = (-2KC\bar{C} - KS\bar{S})\partial_x + (-2K^2yS\bar{C} + KyS\bar{S} + 2K^2y\bar{S}C)\partial_y + (4K^3yC\bar{C} - 2K^2\dot{y}\bar{S}C + 2K^2\dot{y}S\bar{C} + 3K^3yS\bar{S})\partial_{\dot{y}} + (2K^4yS\bar{C} - K^3\dot{y}S\bar{S} - 2K^4y\bar{S}C)\partial_{\ddot{y}}$$

$$[u_2'', u_{15}''] = (\dot{y}C + KyS)\partial_x + (Ky\dot{y}S - 3K^2y^2C)\partial_y + (-3K^2y\dot{y}C + 3K\dot{y}^2S + 2K^3y^2S)\partial_{\dot{y}} + (-K^3y\dot{y}S + 3K^4y^2C)\partial_{\ddot{y}}$$

$$[u_5'', u_9''] = (2K^2y^2\bar{S}C - 2Ky\dot{y}S^2 - 2Ky\dot{y}C^2)\partial_x + (2K^3y^3\bar{S}^2 + 2K^3y^3C^2 + 2K^2y^2\dot{y}\bar{S}C)\partial_y + (-2K^4y^3\bar{S}C + 4K^3y^2\dot{y}S^2 + 2K\dot{y}^3\bar{S}^2 + 4K^3y^2\dot{y}C^2 + 2K\dot{y}C^2)\partial_{\dot{y}} + (-2K^5y^3\bar{S}^2 - 2K^5y^3C^2 - 2K^4y^2\dot{y}\bar{S}C)\partial_{\ddot{y}}$$

$$[u_5'', u_{10}''] = (2K^2y^2\bar{S}^2)\partial_x + (2K^2y^2\dot{y}\bar{S}^2)\partial_y + (-2K^4y^3\bar{S}^2)\partial_{\dot{y}} + (-2K^4y^2\dot{y}\bar{S}^2)\partial_{\ddot{y}}$$

$$[u''_5, u''_{11}] = (-K\dot{y}S\bar{S} - 3K\dot{y}C\bar{C} - 2K^2y\bar{S}C)\partial_x + (K^3y^2S\bar{S} + 2K^2y\dot{y}\bar{S}C - K^3y^2C\bar{C})\partial_y \\ + (-3K^3y\dot{y}S\bar{S} + K^4y^2\bar{S}C - K^2\dot{y}^2\bar{S}C - K^2\dot{y}^2S\bar{C} + 3K^3y\dot{y}C\bar{C} + 3K^4y^2S\bar{C})\partial_{\dot{y}} \\ + (-K^5y^2S\bar{S} - 2K^4y\dot{y}\bar{S}C + K^5y^2C\bar{C})\partial_{\ddot{y}}$$

$$[u''_5, u''_{12}] = (K\dot{y}\bar{S}C - 3K\dot{y}S\bar{C} - 2K^2yS\bar{S})\partial_x + (-K^3y^2\bar{S}C + 2K^2y\dot{y}S\bar{S} - K^3y^2S\bar{C})\partial_y \\ + (3K^3y\dot{y}\bar{S}C + K^4y^2S\bar{S} - K^2\dot{y}^2S\bar{S} + K^2\dot{y}^2C\bar{C} + 3K^3y\dot{y}S\bar{C} - 3K^4y^2C\bar{C})\partial_{\dot{y}} \\ + (K^5y^2\bar{S}C - 2K^4y\dot{y}S\bar{S} + K^5y^2S\bar{C})\partial_{\ddot{y}}$$

$$[u''_5, u''_{13}] = (4K^2y\bar{S}\bar{C} - 3K\dot{y}\bar{S}\bar{S} - 2K^2y\bar{S}\bar{C} - 3K\dot{y}\bar{C}\bar{C})\partial_x + (K^3y^2S\bar{S} \\ + K^3y^2\bar{C}\bar{C} + 2K^2y\dot{y}\bar{S}\bar{C})\partial_y + (-K^4y^2\bar{S}\bar{C} + K^3y\dot{y}\bar{S}\bar{S} + K^2\dot{y}^2\bar{S}\bar{C} \\ - K^4y^2\bar{S}\bar{C} - K^2\dot{y}^2\bar{S}\bar{C} + K^3y\dot{y}\bar{C}\bar{C})\partial_{\dot{y}} + (-K^5y^2\bar{S}\bar{S} - K^5y^2\bar{C}\bar{C} \\ - 2K^4y\dot{y}\bar{S}\bar{C})\partial_{\ddot{y}}$$

$$[u''_5, u''_{14}] = (4K^2y\bar{S}\bar{S} + 3K\dot{y}\bar{S}\bar{C} + 2K^2y\bar{C}\bar{C} - 3K\dot{y}\bar{S}\bar{C})\partial_x + (-K^3y^2\bar{S}\bar{C} \\ + K^3y^2\bar{S}\bar{C} + 2K^2y\dot{y}\bar{S}\bar{S})\partial_y + (-K^4y^2\bar{S}\bar{S} - K^3y\dot{y}\bar{S}\bar{C} + K^2\dot{y}^2S\bar{S} \\ + K^4y^2\bar{C}\bar{C} + K^2\dot{y}^2\bar{C}\bar{C} + K^3y\dot{y}\bar{S}\bar{C})\partial_{\dot{y}} + (K^5y^2\bar{S}\bar{C} - K^5y^2\bar{S}\bar{C} \\ - 2K^4y\dot{y}\bar{S}\bar{S})\partial_{\ddot{y}}$$

$$[u''_5, u''_{15}] = (-2K\dot{y}\dot{y}\bar{C} - 2K^2y^2\bar{S})\partial_x + (2K^2y^2\dot{y}\bar{S} - 2K^3y^3\bar{C})\partial_y + (4K^2y\dot{y}^2 \\ + 2K^4y^3\bar{S} + 2K\dot{y}^3\bar{C})\partial_{\dot{y}} + (-2K^4y^2\dot{y}\bar{S} + 2K^5y^3\bar{C})\partial_{\ddot{y}}$$

$$[u''_6, u''_9] = (2K^2y^2\bar{C}^2)\partial_x + (2K^2y^2\dot{y}\bar{C}^2)\partial_y + (-2K^4y^3\bar{C}^2)\partial_{\dot{y}} + (-2K^4y^2\dot{y}\bar{C}^2)\partial_{\ddot{y}}$$

$$[u''_6, u''_{10}] = (2K^2y^2\bar{S}\bar{C} + 2K\dot{y}\dot{y}\bar{C}^2 + 2K\dot{y}\dot{y}\bar{S}^2)\partial_x + (-2K^3y^3\bar{C}^2 - 2K^3y^3\bar{S}^2 \\ + 2K^2y^2\dot{y}\bar{S}\bar{C})\partial_y + (-2K^4y^3\bar{S}\bar{C} - 4K^3y^2\dot{y}\bar{C}^2 - 2K\dot{y}^3\bar{C}^2 - 4K^3y^2\dot{y}\bar{S}^2 \\ - 2K\dot{y}^3\bar{S}^2)\partial_{\dot{y}} + (2K^5y^3\bar{C}^2 + 2K^5y^3\bar{S}^2 - 2K^4y^2\dot{y}\bar{S}\bar{C})\partial_{\ddot{y}}$$

$$[u''_6, u''_{11}] = (-K\dot{y}S\bar{C} + 3K\dot{y}\bar{S}C - 2K^2yC\bar{C})\partial_x + (K^3y^2S\bar{C} + 2K^2y\dot{y}C\bar{C} + K^3y^2\bar{S}C)\partial_y \\ + (-3K^3y\dot{y}S\bar{C} + K^4y^2C\bar{C} - K^2\dot{y}^2C\bar{C} + K^2\dot{y}^2S\bar{S} - 3K^3y\dot{y}\bar{S}C \\ - 3K^4y^2S\bar{S})\partial_{\dot{y}} + (-K^5y^2S\bar{C} - 2K^4y\dot{y}C\bar{C} - K^5y^2\bar{S}C)\partial_{\ddot{y}}$$

$$[u''_6, u''_{12}] = (K\dot{y}C\bar{C} + 3K\dot{y}S\bar{S} - 2K^2yS\bar{C})\partial_x + (-K^3y^2C\bar{C} + 2K^2y\dot{y}S\bar{C} \\ + K^3y^2S\bar{S})\partial_y + (3K^3y\dot{y}C\bar{C} + K^4y^2S\bar{C} - K^2\dot{y}^2S\bar{C} - K^2\dot{y}^2\bar{S}C)$$

$$\begin{aligned}
 & -3K^3 y \dot{y} S \bar{S} + 3K^4 y^2 \dot{S} \bar{C}) \partial \dot{y} + (K^5 y^2 \dot{C} \bar{C} - 2K^4 y \dot{y} S \bar{C} - K^5 y^2 S \bar{S}) \partial \ddot{y} \\
 [u''_6, u''_{13}] = & (4K^2 y \dot{C} \bar{C} - 3K y \dot{S} \bar{C} + 2K^2 y \dot{S} \bar{S} + 3K y \dot{S} \bar{C}) \partial x + (K^3 y^2 \dot{S} \bar{C} \\
 & - K^3 y^2 \dot{S} \bar{C} + 2K^2 y \dot{y} \dot{C} \bar{C}) \partial y + (-K^4 y^2 \dot{C} \bar{C} + K^3 y \dot{y} \dot{S} \bar{C} + K^2 y^2 \dot{C} \bar{C} \\
 & + K^4 y^2 \dot{S} \bar{S} + K^2 y^2 \dot{S} \bar{S} - K^3 y \dot{y} \dot{S} \bar{C}) \partial \dot{y} + (-K^5 y^2 \dot{S} \bar{C} + K^5 y^2 \dot{S} \bar{C} \\
 & - 2K^4 y \dot{y} \dot{C} \bar{C}) \partial \ddot{y}
 \end{aligned}$$

$$\begin{aligned}
 [u''_6, u''_{14}] = & (4K^2 y \dot{S} \bar{C} + 3K y \dot{C} \bar{C} - 2K^2 y \dot{S} \bar{C} + 3K y \dot{S} \bar{S}) \partial x + (-K^3 y^2 \dot{C} \bar{C} \\
 & - K^3 y^2 \dot{S} \bar{S} + 2K^2 y \dot{y} \dot{S} \bar{C}) \partial y + (-K^4 y^2 \dot{S} \bar{C} - K^3 y \dot{y} \dot{C} \bar{C} + K^2 y^2 \dot{S} \bar{C} \\
 & - K^4 y^2 \dot{S} \bar{C} - K^2 y^2 \dot{S} \bar{C} - K^3 y \dot{y} \dot{S} \bar{S}) \partial \dot{y} + (K^5 y^2 \dot{C} \bar{C} + K^5 y^2 \dot{S} \bar{S} \\
 & - 2K^4 y \dot{y} \dot{S} \bar{C}) \partial \ddot{y}
 \end{aligned}$$

$$\begin{aligned}
 [u''_6, u''_{15}] = & (2K y \dot{y} \dot{S} - 2K^2 y^2 \dot{C}) \partial x + (2K^2 y^2 \dot{y} \dot{C} + 2K^3 y^3 \dot{S}) \partial y + (4K^2 y \dot{y}^2 \dot{C} \\
 & + 2K^4 y^3 \dot{C} - 2K y \dot{y}^3 \dot{S}) \partial \dot{y} + (-2K^4 y^2 \dot{y} \dot{C} - 2K^5 y^3 \dot{S}) \partial \ddot{y}
 \end{aligned}$$

$$\begin{aligned}
 [u''_7, u''_9] = & (2K^2 y^3 \dot{C} \bar{C} - 2K y^2 \dot{y} \dot{S} \bar{C} - 2K^2 y^3 S \bar{S} - K y^2 \dot{y} S \bar{S} - y \dot{y}^2 \dot{C} \bar{C}) \partial x \\
 & - K^3 y^4 \dot{S} \bar{C} + K^2 y^3 \dot{y} \dot{C} \bar{C}) \partial y + (-2K^4 y^4 \dot{C} \bar{C} + y^4 \dot{C} \bar{C} + 2K y \dot{y}^3 \dot{S} \bar{C} \\
 & + 2K^4 y^4 S \bar{S} + 2K^3 y^3 \dot{y} \dot{S} \bar{C} + 2K^2 y^2 \dot{y}^2 S \bar{S} + K y \dot{y}^3 \dot{S} \bar{C}) \partial \dot{y} \\
 & (K^5 y^4 \dot{S} \bar{C} - K^4 y^3 \dot{y} \dot{C} \bar{C}) \partial \ddot{y}.
 \end{aligned}$$

$$\begin{aligned}
 [u''_7, u''_{10}] = & (2K^2 y^3 \dot{S} \bar{C} + 2K y^2 \dot{y} \dot{C} \bar{C} + 2K^2 y^3 \dot{S} \bar{C} - K y^2 \dot{y} S \bar{S} - y \dot{y}^2 \dot{S} \bar{C}) \partial x \\
 & + (-K^3 y^4 \dot{S} \bar{S} + K^2 y^3 \dot{y} \dot{S} \bar{C}) \partial y + (-2K^4 y^4 \dot{S} \bar{C} + y^4 \dot{S} \bar{C} - 2K y \dot{y}^3 \dot{C} \bar{C} \\
 & - 2K^4 y^4 \dot{S} \bar{C} - 2K^3 y^3 \dot{y} \dot{C} \bar{C} - 2K^2 y^2 \dot{y}^2 \dot{S} \bar{C} + K y \dot{y}^3 \dot{S} \bar{S}) \partial \dot{y} \\
 & + (K^5 y^4 \dot{S} \bar{S} - K^4 y^3 \dot{y} \dot{S} \bar{C}) \partial \ddot{y}
 \end{aligned}$$

$$\begin{aligned}
 [u''_7, u''_{11}] = & (-K y \dot{y} \dot{S} \bar{C} - y^2 \dot{C} \bar{C}) \partial x + (K^3 y^3 \dot{S} \bar{C} + K^2 y^2 \dot{y} \dot{C} \bar{C}) \partial y + (-K y^3 \dot{S} \bar{C} \\
 & + K^2 y \dot{y}^2 \dot{C} \bar{C} - K^4 y^3 \dot{S} \bar{S} - K^2 y \dot{y}^2 \dot{S} \bar{S}) \partial \dot{y} + (-K^5 y^3 \dot{S} \bar{C} - K^4 y^2 \dot{y} \dot{C} \bar{C}) \partial \ddot{y}
 \end{aligned}$$

$$\begin{aligned}
 [u''_7, u''_{12}] = & (K y \dot{y} \dot{C} \bar{C} - y^2 \dot{S} \bar{C}) \partial x + (-K^3 y^3 \dot{C} \bar{C} + K^2 y^2 \dot{y} \dot{S} \bar{C}) \partial y + (-K y^2 \dot{y} \dot{S} \bar{C} \\
 & + K^4 y^3 \dot{S} \bar{C} + 2K^2 y \dot{y}^2 \dot{S} \bar{C} - K^3 y^2 \dot{y} \dot{S} \bar{S} - K y^3 \dot{S} \bar{S}) \partial \dot{y} + (K^5 y^3 \dot{C} \bar{C} - K^4 y^2 \dot{y} \dot{S} \bar{C}) \partial \ddot{y}
 \end{aligned}$$

$$\begin{aligned}
 [u''_7, u''_{13}] = & (4K^2 y^2 \dot{C} \bar{C} - 3K y \dot{y} \dot{S} \bar{C} - y^2 \dot{C} \bar{C}) \partial x + (-K^3 y^3 \dot{S} \bar{C} + K^2 y^2 \dot{y} \dot{C} \bar{C}) \partial y \\
 & + (-4K^4 y^3 \dot{C} \bar{C} + 3K^3 y^2 \dot{y} \dot{S} \bar{C} + K^2 y \dot{y}^2 \dot{C} \bar{C} + K^4 y^3 \dot{S} \bar{S} + 3K^3 y^2 \dot{y} \dot{S} \bar{C}
 \end{aligned}$$

$$-3K^2 y \dot{y}^2 S \bar{S} - K \dot{y}^3 S \bar{C}) \partial \dot{y} + (K^5 y^3 \bar{S} C - K^4 y^2 \dot{y} C \bar{C}) \partial \dot{y}$$

$$[u''_7, u''_{14}] = (4K^2 y^2 \bar{S} C + 3K y \dot{y} C \bar{C} - \dot{y}^2 \bar{S} C) \partial x + (K^3 y^3 C \bar{C} + K^2 y^2 \dot{y} \bar{S} C) \partial y \\ + (-4K^4 y^3 \bar{S} C - 3K^3 y^2 \dot{y} C \bar{C} + K^2 y \dot{y}^2 \bar{S} C - K^4 y^3 S \bar{C} + 3K^3 y^2 \dot{y} S \bar{S} + 3K^2 y \dot{y}^2 S \bar{C} \\ - K \dot{y}^3 S \bar{S}) \partial \dot{y} + (-K^5 y^3 C \bar{C} - K^4 y^2 \dot{y} \bar{S} C) \partial \ddot{y}$$

$$[u''_7, u''_{15}] = (-K y^2 \dot{y} S - y \dot{y}^2 C) \partial x + (K^2 y^3 \dot{y} C + K^3 y^4 S) \partial y + (2K^3 y^3 \dot{y} S + K y \dot{y}^3 S \\ + 2K^2 y^2 \dot{y}^2 C + \dot{y}^4 C) \partial \dot{y} + (-K^4 y^3 \dot{y} C - K^5 y^4 S) \partial \ddot{y}$$

$$[u''_8, u''_9] = (2K^2 y^3 S \bar{C} - 2K y^2 \dot{y} S \bar{S} + 2K^2 y^3 \bar{S} C + K y^2 \dot{y} C \bar{C} - y \dot{y}^2 S \bar{C}) \partial x \\ + (K^3 y^4 C \bar{C} + K^2 y^3 \dot{y} S \bar{C}) \partial y + (-2K^4 y^4 S \bar{C} + \dot{y}^4 S \bar{C} + 2K y \dot{y}^3 S \bar{S} \\ - 2K^4 y^4 \bar{S} C + 2K^3 y^3 \dot{y} S \bar{S} - 2K^2 y^2 \dot{y}^2 \bar{S} C - K y \dot{y}^3 C \bar{C}) \partial \dot{y} + (-K^5 y^4 C \bar{C} \\ - K^4 y^3 \dot{y} S \bar{C}) \partial \ddot{y}$$

$$[u''_8, u''_{10}] = (2K^2 y^3 S \bar{S} + 2K y^2 \dot{y} S \bar{C} - 2K^2 y^3 C \bar{C} + K y^2 \dot{y} \bar{S} C - y \dot{y}^2 S \bar{S}) \partial x \\ + (K^3 y^4 \bar{S} C + K^2 y^3 \dot{y} S \bar{S}) \partial y + (-2K^4 y^4 S \bar{S} + \dot{y}^4 S \bar{S} - 2K y \dot{y}^3 S \bar{C} \\ + 2K^4 y^4 C \bar{C} - 2K^3 y^3 \dot{y} S \bar{C} + 2K^2 y^2 \dot{y}^2 C \bar{C} - K y \dot{y}^3 \bar{S} C) \partial \dot{y} \\ + (-K^5 y^4 \bar{S} C - K^4 y^3 \dot{y} S \bar{S}) \partial \ddot{y}$$

$$[u''_8, u''_{11}] = (-K y \dot{y} S^2 - \dot{y}^2 S C) \partial x + (K^3 y^3 S^2 + K^2 y^2 \dot{y} S C) \partial y + (K^3 y^2 \dot{y} S^2 \\ + K^4 y^3 S C + 2K^2 y \dot{y}^2 S C + K^3 y^2 \dot{y} C^2 + K \dot{y}^3 C^2) \partial \dot{y} + (-K^5 y^3 S^2 \\ - K^4 y^2 \dot{y} S C) \partial \ddot{y}$$

$$[u''_8, u''_{12}] = (K y \dot{y} S C - \dot{y}^2 S^2) \partial x + (-K^3 y^3 S C + K^2 y^2 \dot{y} S^2) \partial y + (K \dot{y}^3 S C - K^2 y \dot{y}^2 S^2 \\ - K^4 y^3 C^2 - K^2 y \dot{y}^2 C^2) \partial \dot{y} + (K^5 y^3 S C - K^4 y^2 \dot{y} S^2) \partial \ddot{y}$$

$$[u''_8, u''_{13}] = (4K^2 y^2 S \bar{C} - 3K y \dot{y} S \bar{S} - \dot{y}^2 S \bar{C}) \partial x + (-K^3 y^3 S \bar{S} + K^2 y^2 \dot{y} S \bar{C}) \partial y \\ + (-4K^4 y^3 S \bar{C} + 3K^3 y^2 \dot{y} S \bar{S} + K^2 y \dot{y}^2 S \bar{C} - K^4 y^3 \bar{S} C - 3K^3 y^2 \dot{y} C \bar{C} \\ + 3K^2 y \dot{y}^2 \bar{S} C + K \dot{y}^3 C \bar{C}) \partial \dot{y} + (K^5 y^3 S \bar{S} - K^4 y^2 \dot{y} S \bar{C}) \partial \ddot{y}$$

$$[u''_8, u''_{14}] = (4K^2 y^2 S \bar{S} + 3K y \dot{y} S \bar{C} - \dot{y}^2 S \bar{S}) \partial x + (K^3 y^3 S \bar{C} + K^2 y^2 \dot{y} S \bar{S}) \partial y \\ + (-4K^4 y^3 S \bar{S} - 3K^3 y^2 \dot{y} S \bar{C} + K^2 y \dot{y}^2 S \bar{S} + K^4 y^3 C \bar{C} - 3K^3 y^2 \dot{y} \bar{S} C \\ - 3K^2 y \dot{y}^2 C \bar{C} + K \dot{y}^3 \bar{S} C) \partial \dot{y} + (-K^5 y^3 S \bar{C} - K^4 y^2 \dot{y} S \bar{S}) \partial \ddot{y}$$

$$[U''_8, U''_{15}] = (K y^2 \dot{y} C - y \dot{y}^2 S) \partial_x + (K^2 y^3 \dot{y} S - K^3 y^4 C) \partial_y + (-2K^3 y^3 \dot{y} C - K y \dot{y}^3 C + 2K^2 y^2 \dot{y}^2 S + \dot{y}^4 S) \partial_{\dot{y}} + (-K^4 y^3 \dot{y} S + K^5 y^4 C) \partial_{\ddot{y}}$$

$$[U''_9, U''_{10}] = (2K^3 y^4 \bar{S}^2 + 2K^3 y^4 \bar{C}^2) \partial_x + (2K^3 y^4 \dot{y} \bar{S}^2 + 2K^3 y^4 \dot{y} \bar{C}^2) \partial_y + (-2K^5 y^5 \bar{S}^2 - 2K^5 y^5 \bar{C}^2) \partial_{\dot{y}} + (-2K^5 y^4 \dot{y} \bar{S}^2 - 2K^5 y^4 \dot{y} \bar{C}^2) \partial_{\ddot{y}}$$

$$[U''_9, U''_{11}] = (-2K^2 y^2 \dot{y} S \bar{S} - 2K y \dot{y}^2 S \bar{C} - 2K^2 y^2 \dot{y} C \bar{C} - \dot{y}^3 C \bar{C} + 2K^3 y^3 \bar{S} C - K^3 y^3 S \bar{C}) \partial_x + (2K^4 y^4 S \bar{S} + K^3 y^3 \dot{y} S \bar{C} + 2K^3 y^3 \dot{y} \bar{S} C + K^4 y^4 C \bar{C}) \partial_y + (2K^4 y^3 \dot{y} S \bar{S} + 4K^3 y^2 \dot{y}^2 S \bar{C} - 2K^5 y^4 \bar{S} C + 4K^4 y^3 \dot{y} C \bar{C} - 2K^3 y^2 \dot{y}^2 \bar{S} C + 3K^2 y \dot{y}^3 C \bar{C} + 2K^5 y^4 S \bar{C} + K y^4 S \bar{C} - 2K y^4 \bar{S} C) \partial_{\dot{y}} + (-2K^6 y^4 S \bar{S} - K^5 y^3 \dot{y} S \bar{C} - 2K^5 y^3 \dot{y} \bar{S} C - K^6 y^4 C \bar{C}) \partial_{\ddot{y}}$$

$$[U''_9, U''_{12}] = (2K^2 y^2 \dot{y} \bar{S} C + 2K y \dot{y}^2 C \bar{C} - 2K^2 y^2 \dot{y} S \bar{C} - \dot{y}^3 S \bar{C} + 2K^3 y^3 S \bar{S} + K^3 y^3 C \bar{C}) \partial_x + (-2K^4 y^4 \bar{S} C - K^3 y^3 \dot{y} C \bar{C} + 2K^3 y^3 \dot{y} S \bar{S} + K^4 y^4 S \bar{C}) \partial_y + (-2K^4 y^3 \dot{y} \bar{S} C - 4K^3 y^2 \dot{y}^2 C \bar{C} - 2K^5 y^4 S \bar{S} + 4K^4 y^3 \dot{y} S \bar{C} - 2K^3 y^2 \dot{y}^2 S \bar{S} + 3K^2 y \dot{y}^3 S \bar{C} - 2K^5 y^4 C \bar{C} - K y^4 C \bar{C} - 2K y^4 S \bar{S}) \partial_{\dot{y}} + (2K^6 y^4 \bar{S} C + K^5 y^3 \dot{y} C \bar{C} - 2K^5 y^3 \dot{y} S \bar{S} - K^6 y^4 S \bar{C}) \partial_{\ddot{y}}$$

$$[U''_9, U''_{13}] = (6K^3 y^3 \bar{S} \bar{C} + 4K^2 y^2 \dot{y} \bar{C} \bar{C} - 2K^2 y^2 \dot{y} \bar{S} \bar{S} - 4K y \dot{y}^2 \bar{S} \bar{C} - 3K^3 y^3 \bar{S} \bar{C} - \dot{y}^3 \bar{C} \bar{C}) \partial_x + (-2K^4 y^4 \bar{S} \bar{S} - 3K^3 y^3 \dot{y} \bar{S} \bar{C} - K^4 y^4 \bar{C} \bar{C} + 2K^3 y^3 \dot{y} \bar{S} \bar{C}) \partial_y + (-6K^5 y^4 \bar{S} \bar{C} - 8K^4 y^3 \dot{y} \bar{C} \bar{C} + 2K^4 y^3 \dot{y} \bar{S} \bar{S} + 8K^3 y^2 \dot{y}^2 \bar{S} \bar{C} + 2K^3 y^2 \dot{y}^2 \bar{S} \bar{C} + K^2 y \dot{y}^3 \bar{C} \bar{C} + 2K^5 y^4 \bar{S} \bar{C} + K y^4 \bar{S} \bar{C} - 4K^2 y \dot{y}^3 \bar{S} \bar{S} - 2K y^4 \bar{S} \bar{C}) \partial_{\dot{y}} + (2K^6 y^4 \bar{S} \bar{S} + 3K^5 y^3 \dot{y} \bar{S} \bar{C} + K^6 y^4 \bar{C} \bar{C} - 2K^5 y^3 \dot{y} \bar{S} \bar{C}) \partial_{\ddot{y}}$$

$$[U''_9, U''_{14}] = (6K^3 y^3 \bar{S} \bar{S} + 4K^2 y^2 \dot{y} \bar{S} \bar{C} + 2K^2 y^2 \dot{y} \bar{S} \bar{C} + 4K y \dot{y}^2 \bar{C} \bar{C} + 3K^3 y^3 \bar{C} \bar{C} - \dot{y}^3 \bar{S} \bar{C}) \partial_x + (2K^4 y^4 \bar{S} \bar{C} + 3K^3 y^3 \dot{y} \bar{C} \bar{C} - K^4 y^4 \bar{S} \bar{C} + 2K^3 y^3 \dot{y} \bar{S} \bar{S}) \partial_y + (-6K^5 y^4 \bar{S} \bar{S} - 8K^4 y^3 \dot{y} \bar{S} \bar{C} - 2K^4 y^3 \dot{y} \bar{S} \bar{C} - 8K^3 y^2 \dot{y}^2 \bar{C} \bar{C} + 2K^3 y^2 \dot{y}^2 \bar{S} \bar{S} + K^2 y \dot{y}^3 \bar{S} \bar{C} - 2K^5 y^4 \bar{C} \bar{C} - K y^4 \bar{C} \bar{C} + 4K^2 y \dot{y}^3 \bar{S} \bar{C} - 2K y^4 \bar{S} \bar{S}) \partial_{\dot{y}} + (-2K^6 y^4 \bar{S} \bar{C} - 3K^5 y^3 \dot{y} \bar{C} \bar{C} + K^6 y^4 \bar{S} \bar{C} - 2K^5 y^3 \dot{y} \bar{S} \bar{S}) \partial_{\ddot{y}}$$

$$[U''_9, U''_{15}] = (2K^3 y^4 \bar{S}) \partial_x + (2K^3 y^4 \dot{y} \bar{S}) \partial_y + (-2K^5 y^5 \bar{S}) \partial_{\dot{y}} + (-2K^5 y^4 \dot{y} \bar{S}) \partial_{\ddot{y}}$$

$$[U''_{10}, U''_{11}] = (2K^2 y^2 \dot{y} S \bar{C} - 2K y \dot{y}^2 S \bar{S} - 2K^2 y^2 \dot{y} \bar{S} C - \dot{y}^3 \bar{S} C - 2K^3 y^3 C \bar{C} - K^3 y^3 S \bar{S}) \partial_x + (-2K^4 y^4 S \bar{C} + K^3 y^3 \dot{y} S \bar{S} - 2K^3 y^3 \dot{y} C \bar{C} + K^4 y^4 \bar{S} C) \partial_y + (-2K^4 y^3 \dot{y} S \bar{C} + 4K^3 y^2 \dot{y}^2 S \bar{S} + 2K^5 y^4 C \bar{C} + 4K^4 y^3 \dot{y} \bar{S} C + 2K^3 y^2 \dot{y}^2 C \bar{C} + 3K^2 y \dot{y}^3 \bar{S} C + 2K^5 y^4 S \bar{S} + K y^4 \dot{y} S \bar{S} + 2K y^4 C \bar{C}) \partial_{\dot{y}} + (2K^6 y^4 S \bar{C} - K^5 y^3 \dot{y} S \bar{S} + 2K^5 y^3 \dot{y} C \bar{C} - K^6 y^4 \bar{S} C) \partial_{\ddot{y}}$$

$$[U''_{10}, U''_{12}] = (-2K^2 y^2 \dot{y} C \bar{C} + 2K y \dot{y}^2 \bar{S} C - 2K^2 y^2 \dot{y} S \bar{S} - \dot{y}^3 S \bar{S} - 2K^3 y^3 S \bar{C} + K^3 y^3 \bar{S} C) \partial_x + (2K^4 y^4 C \bar{C} - K^3 y^3 \dot{y} \bar{S} C - 2K^3 y^3 \dot{y} S \bar{C} + K^4 y^4 S \bar{S}) \partial_y + (2K^4 y^3 \dot{y} C \bar{C} - 4K^3 y^2 \dot{y}^2 \bar{S} C + 2K^5 y^4 S \bar{C} + 4K^4 y^3 \dot{y} S \bar{S} + 2K^3 y^2 \dot{y}^2 S \bar{C} + 3K^2 y \dot{y}^3 S \bar{S} - 2K^5 y^4 \bar{S} C - K y^4 \dot{y} \bar{S} C + 2K y^4 S \bar{C}) \partial_{\dot{y}} + (-2K^6 y^4 C \bar{C} + K^5 y^3 \dot{y} \bar{S} C + 2K^5 y^3 \dot{y} S \bar{C} - K^6 y^4 S \bar{S}) \partial_{\ddot{y}}$$

$$[U''_{10}, U''_{13}] = (-6K^3 y^3 \bar{C} \bar{C} + 4K^2 y^2 \dot{y} \bar{C} \bar{S} + 2K^2 y^2 \dot{y} \bar{C} \bar{S} - 4K y \dot{y}^2 \bar{S} \bar{S} - 3K^3 y^3 \bar{S} \bar{S} - \dot{y}^3 \bar{C} \bar{S}) \partial_x + (2K^4 y^4 \bar{C} \bar{S} - 3K^3 y^3 \dot{y} \bar{S} \bar{S} - K^4 y^4 \bar{C} \bar{S} - 2K^3 y^3 \dot{y} \bar{C} \bar{C}) \partial_y + (6K^5 y^4 \bar{C} \bar{C} - 8K^4 y^3 \dot{y} \bar{C} \bar{S} - 2K^4 y^3 \dot{y} \bar{C} \bar{S} + 8K^3 y^2 \dot{y}^2 \bar{S} \bar{S} - 2K^3 y^2 \dot{y}^2 \bar{C} \bar{C} + K^2 y \dot{y}^3 \bar{C} \bar{S} + 2K^5 y^4 \bar{S} \bar{S} + K y^4 \dot{y} \bar{S} \bar{S} + 4K^2 y \dot{y}^3 \bar{C} \bar{S} + 2K y^4 \bar{C} \bar{C}) \partial_{\dot{y}} + (-2K^6 y^4 \bar{C} \bar{S} + 3K^5 y^3 \dot{y} \bar{S} \bar{S} + K^6 y^4 \bar{C} \bar{S} + 2K^5 y^3 \dot{y} \bar{C} \bar{C}) \partial_{\ddot{y}}$$

$$[U''_{10}, U''_{14}] = (-6K^3 y^3 \bar{C} \bar{S} + 4K^2 y^2 \dot{y} \bar{S} \bar{S} - 2K^2 y^2 \dot{y} \bar{C} \bar{C} + 4K y \dot{y}^2 \bar{C} \bar{S} + 3K^3 y^3 \bar{C} \bar{S} - \dot{y}^3 \bar{S} \bar{S}) \partial_x + (-2K^4 y^4 \bar{C} \bar{C} + 3K^3 y^3 \dot{y} \bar{C} \bar{S} - K^4 y^4 \bar{S} \bar{S} - 2K^3 y^3 \dot{y} \bar{C} \bar{S}) \partial_y + (6K^5 y^4 \bar{C} \bar{S} - 8K^4 y^3 \dot{y} \bar{S} \bar{S} + 2K^4 y^3 \dot{y} \bar{C} \bar{C} - 8K^3 y^2 \dot{y}^2 \bar{C} \bar{S} - 2K^3 y^2 \dot{y}^2 \bar{C} \bar{S} + K^2 y \dot{y}^3 \bar{S} \bar{S} - 2K^5 y^4 \bar{C} \bar{S} - K y^4 \dot{y} \bar{C} \bar{S} - 4K^2 y \dot{y}^3 \bar{C} \bar{C} + 2K y^4 \bar{C} \bar{S}) \partial_{\dot{y}} + (2K^6 y^4 \bar{C} \bar{C} - 3K^5 y^3 \dot{y} \bar{C} \bar{S} + K^6 y^4 \bar{S} \bar{S} + 2K^5 y^3 \dot{y} \bar{C} \bar{S}) \partial_{\ddot{y}}$$

$$[U''_{10}, U''_{15}] = (-2K^3 y^4 \bar{C}) \partial_x + (-2K^3 y^4 \dot{y} \bar{C}) \partial_y + (2K^5 y^5 \bar{C}) \partial_{\dot{y}} + (2K^5 y^4 \dot{y} \bar{C}) \partial_{\ddot{y}}$$

$$[U''_{11}, U''_{12}] = (2K y^2 C^2 + 2K y^2 S^2 + K^3 y^2 C^2 + K^3 y^2 S^2) \partial_x + (-K^3 y^2 \dot{y} C^2 - K^3 y^2 \dot{y} S^2) \partial_y + (K^5 y^3 S^2 + K^5 y^3 C^2) \partial_{\dot{y}} + (K^5 y^2 \dot{y} C^2 + K^5 y^2 \dot{y} S^2) \partial_{\ddot{y}}$$

$$[U''_{11}, U''_{13}] = (8K^2 y \dot{y} \bar{C} \bar{C} - 4K y^2 \bar{C} \bar{S} + 2K^2 y \dot{y} \bar{S} \bar{S} + 2K y^2 \bar{C} \bar{S} + K^3 y^2 \bar{C} \bar{S})$$

$$+K^3 y^2 \bar{C} \bar{S}) \partial_x + (-3K^3 y^2 \bar{y} \bar{C} \bar{S} - 2K^4 y^3 \bar{S} \bar{S} - K^3 y^2 \bar{y} \bar{C} \bar{S}) \partial_y + (-6K^4 y^2 \bar{y} \bar{C} \bar{C} + 6K^3 y \bar{y}^2 \bar{C} \bar{S} + 2K^2 \bar{y}^3 \bar{C} \bar{C} + 6K^4 y^2 \bar{y} \bar{S} \bar{S} + 2K^3 y \bar{y}^2 \bar{C} \bar{S} + K^5 y^3 \bar{C} \bar{S} - 5K^5 y^3 \bar{C} \bar{S}) \partial_{\bar{y}} + (3K^5 y^2 \bar{y} \bar{C} \bar{S} + 2K^6 y^3 \bar{S} \bar{S} + K^5 y^2 \bar{y} \bar{C} \bar{S}) \partial_{\bar{y}}$$

$$[U''_{11}, U''_{14}] = (8K^2 y \bar{y} \bar{C} \bar{S} + 4K \bar{y}^2 \bar{C} \bar{C} - 2K^2 y \bar{y} \bar{C} \bar{S} + 2K \bar{y}^2 \bar{S} \bar{S} - K^3 y^2 \bar{C} \bar{C} + K^3 y^2 \bar{S} \bar{S}) \partial_x + (3K^3 y^2 \bar{y} \bar{C} \bar{C} + 2K^4 y^3 \bar{C} \bar{S} - K^3 y^2 \bar{y} \bar{S} \bar{S}) \partial_y + (-6K^4 y^2 \bar{y} \bar{C} \bar{S} - 6K^3 y \bar{y}^2 \bar{C} \bar{C} + 2K^2 \bar{y}^3 \bar{C} \bar{S} - 6K^4 y^2 \bar{y} \bar{C} \bar{S} + 2K^3 y \bar{y}^2 \bar{S} \bar{S} - K^5 y^3 \bar{C} \bar{C} - 5K^5 y^3 \bar{S} \bar{S}) \partial_{\bar{y}} + (-3K^5 y^2 \bar{y} \bar{C} \bar{C} - 2K^6 y^3 \bar{C} \bar{S} + K^5 y^2 \bar{y} \bar{S} \bar{S}) \partial_{\bar{y}}$$

$$[U''_{11}, U''_{15}] = (2K y \bar{y}^2 \bar{S} + K^3 y^3 \bar{S} + \bar{y}^3 \bar{C}) \partial_x + (-K^3 y^3 \bar{y} \bar{S} + K^4 y^4 \bar{C}) \partial_y + (-2K^3 y^2 \bar{y}^2 \bar{S} + 2K^4 y^3 \bar{y} \bar{C} + K^2 y \bar{y}^3 \bar{C} - K y^4 \bar{S}) \partial_{\bar{y}} + (K^5 y^3 \bar{y} \bar{S} - K^6 y^4 \bar{C}) \partial_{\bar{y}}$$

$$[U''_{12}, U''_{13}] = (8K^2 y \bar{y} \bar{C} \bar{S} - 4K \bar{y}^2 \bar{S} \bar{S} - 2K^2 y \bar{y} \bar{C} \bar{S} - 2K \bar{y}^2 \bar{C} \bar{C} + K^3 y^2 \bar{S} \bar{S} - K^3 y^2 \bar{C} \bar{C}) \partial_x + (-3K^3 y^2 \bar{y} \bar{S} \bar{S} + 2K^4 y^3 \bar{C} \bar{S} + K^3 y^2 \bar{y} \bar{C} \bar{C}) \partial_y + (-6K^4 y^2 \bar{y} \bar{C} \bar{S} + 6K^3 y \bar{y}^2 \bar{S} \bar{S} + 2K^2 \bar{y}^3 \bar{C} \bar{S} - 6K^4 y^2 \bar{y} \bar{C} \bar{S} - 2K^3 y \bar{y}^2 \bar{C} \bar{C} + K^5 y^3 \bar{S} \bar{S} + 5K^5 y^3 \bar{C} \bar{C}) \partial_{\bar{y}} + (3K^5 y^2 \bar{y} \bar{S} \bar{S} - 2K^6 y^3 \bar{C} \bar{S} - K^5 y^2 \bar{y} \bar{C} \bar{C}) \partial_{\bar{y}}$$

$$[U''_{12}, U''_{14}] = (8K^2 y \bar{y} \bar{S} \bar{S} - 4K \bar{y}^2 \bar{C} \bar{S} + 2K^2 y \bar{y} \bar{C} \bar{C} - 2K \bar{y}^2 \bar{C} \bar{S} - K^3 y^2 \bar{C} \bar{S} - K^3 y^2 \bar{C} \bar{S}) \partial_x + (3K^3 y^2 \bar{y} \bar{C} \bar{S} - 2K^4 y^3 \bar{C} \bar{C} + K^3 y^2 \bar{y} \bar{C} \bar{S}) \partial_y + (-6K^4 y^2 \bar{y} \bar{S} \bar{S} - 6K^3 y \bar{y}^2 \bar{C} \bar{S} + 2K^2 \bar{y}^3 \bar{S} \bar{S} + 6K^4 y^2 \bar{y} \bar{C} \bar{C} - 2K^3 y \bar{y}^2 \bar{C} \bar{S} - K^5 y^3 \bar{C} \bar{S} + 5K^5 y^3 \bar{C} \bar{S}) \partial_{\bar{y}} + (-3K^5 y^2 \bar{y} \bar{C} \bar{S} + 2K^6 y^3 \bar{C} \bar{C} - K^5 y^2 \bar{y} \bar{C} \bar{S}) \partial_{\bar{y}}$$

$$[U''_{12}, U''_{15}] = (-2K y \bar{y}^2 \bar{C} - K^3 y^3 \bar{C} + \bar{y}^3 \bar{S}) \partial_x + (K^3 y^3 \bar{y} \bar{C} + K^4 y^4 \bar{S}) \partial_y + (2K^3 y^2 \bar{y}^2 \bar{C} + 2K^4 y^3 \bar{y} \bar{S} + K^2 y \bar{y}^3 \bar{S} + K y^4 \bar{C}) \partial_{\bar{y}} + (-K^5 y^3 \bar{y} \bar{C} - K^6 y^4 \bar{S}) \partial_{\bar{y}}$$

$$[U''_{13}, U''_{14}] = (7K^3 y^2 \bar{S}^2 + 7K^3 y^2 \bar{C}^2 + 4K \bar{y}^2 \bar{S}^2 + 4K \bar{y}^2 \bar{C}^2) \partial_x + (3K^3 y^2 \bar{y} \bar{S}^2 + 3K^3 y^2 \bar{y} \bar{C}^2) \partial_y + (-3K^5 y^3 \bar{S}^2 - 3K^5 y^3 \bar{C}^2) \partial_{\bar{y}} + (-3K^5 y^2 \bar{y} \bar{S}^2 - 3K^5 y^2 \bar{y} \bar{C}^2) \partial_{\bar{y}}$$

$$[U''_{13}, U''_{15}] = (-6K^2 y^2 \bar{y} \bar{C} + 4K y \bar{y}^2 \bar{S} + \bar{y}^3 \bar{C} - K^3 y^3 \bar{S}) \partial_x + (3K^3 y^3 \bar{y} \bar{S} - K^4 y^4 \bar{C}) \partial_y + (2K^4 y^3 \bar{y} \bar{C} + 2K^3 y^2 \bar{y}^2 \bar{S} + 3K^2 y \bar{y}^3 \bar{C} - K y^4 \bar{S}) \partial_{\bar{y}}$$

$$+(-3K^5y^3\dot{y}\bar{S}+K^6y^4\bar{C})\partial\ddot{y}$$

$$\begin{aligned} [u''_{14}, u''_{15}] = & (-6K^2y^2\dot{y}\bar{S}-4Ky\dot{y}^2\bar{C}+\dot{y}^3\bar{S}+K^3y^3\bar{C})\partial_x + (-3K^3y^3\dot{y}\bar{C}-K^4y^4\bar{S})\partial_y \\ & + (2K^4y^3\dot{y}\bar{S}-2K^3y^2\dot{y}^2\bar{C}+3K^2y\dot{y}^3\bar{S}+Ky^4\bar{C})\partial\dot{y} + (3K^5y^3\dot{y}\bar{C} \\ & + K^6y^4\bar{S})\partial\ddot{y} \end{aligned}$$

APPENDIX C: Asterisked Commutators From Table J

$$[X_1, X_8] = (2x^3) \partial x + (-6x^3 \dot{y} + 4x^4 \ddot{y} + 8x^2 y) \partial y + (-16x^2 \dot{y} + 10x^3 \ddot{y} + 16x \dot{y}) \partial \dot{y} \\ + (8x^2 \ddot{y} - 16x \dot{y} + 16y) \partial \ddot{y}$$

$$[X_1, X_{10}] = (-2x \dot{y}) \partial y + (-4 \dot{y} - 2x \ddot{y}) \partial \dot{y} + (-8 \ddot{y}) \partial \ddot{y}$$

$$[X_1, X_{11}] = (2x) \partial x + (-2x^2 \ddot{y}) \partial y + (-4x \ddot{y} - 2 \dot{y}) \partial \dot{y} + (-8 \ddot{y}) \partial \ddot{y}$$

$$[X_1, X_{12}] = (2) \partial x + (-2x \ddot{y}) \partial y + (-2 \ddot{y}) \partial \dot{y}$$

$$[X_1, X_{13}] = (4x^3) \partial x + (4x^2 y + 2x^4 \ddot{y}) \partial y + (8xy + 8x^3 \ddot{y} - 8x^2 \dot{y}) \partial \dot{y} \\ + (4x^2 \ddot{y} - 8x \dot{y} + 8y) \partial \ddot{y}$$

$$[X_1, X_{14}] = (-4xy + 2x^3 \ddot{y}) \partial y + (-4y + 6x^2 \ddot{y} - 4x \dot{y}) \partial \dot{y} + (-8 \dot{y} + 8x \ddot{y}) \partial \ddot{y}$$

$$[X_2, X_8] = (-4x^2 \dot{y} + 2x^4 \ddot{y} + 6x^2 y) \partial x + (-6x^2 y \dot{y} + 4x^3 y \ddot{y} + 8xy^2) \partial y \\ + (-8xy \dot{y} + 6x^2 y \ddot{y} + 8y^2) \partial \dot{y} + (12x^2 \dot{y} \ddot{y} - 4x^3 \ddot{y}^2 - 8x \dot{y}^2 \\ - 8xy \ddot{y} + 8y \dot{y}) \partial \ddot{y}$$

$$[X_2, X_9] = (-3x^2 \dot{y} + 2x^3 \ddot{y} + 2xy) \partial x + (-2xy \dot{y} + 2x^2 y \ddot{y}) \partial y + (-x^2 y \ddot{y} - 4y \dot{y} \\ + 2xy \ddot{y} + 2x \dot{y}^2) \partial \dot{y} + (-4x^2 \ddot{y}^2 + 8x \dot{y} \ddot{y} - 2 \dot{y}^2 - 4y \ddot{y}) \partial \ddot{y}$$

$$[X_2, X_{10}] = (-2x \dot{y} + 2y) \partial x + (-2y \dot{y}) \partial y + (-2y \ddot{y} - 2 \dot{y}^2 + 2x \dot{y} \ddot{y}) \partial \dot{y} + (-4 \dot{y} \ddot{y} + 4x \ddot{y}^2) \partial \ddot{y}$$

$$[X_2, X_{11}] = (-3x^2 \ddot{y} + 2x \dot{y}) \partial x + (-2xy \ddot{y}) \partial y + (-2y \ddot{y} - 2 \dot{y}^2 + 2x \dot{y} \ddot{y}) \partial \dot{y} + (6x \ddot{y}^2 - 6 \dot{y} \ddot{y}) \partial \ddot{y}$$

$$[X_2, X_{12}] = (-4x \ddot{y} + 2 \dot{y}) \partial x + (-2y \ddot{y}) \partial y + (2 \ddot{y}^2) \partial \ddot{y}$$

$$[X_2, X_{13}] = (6x^2 y + x^4 \ddot{y} - 2x^3 \dot{y}) \partial x + (4xy^2 + 2x^3 y \ddot{y}) \partial y + (-4xy \dot{y} + 6x^2 y \ddot{y} + 4y^2) \partial \dot{y} \\ + (6x^2 \dot{y} \ddot{y} - 2x^3 \ddot{y}^2 - 4xy \ddot{y} - 4x \dot{y}^2 + 4y \dot{y}) \partial \ddot{y}$$

$$[X_2, X_{14}] = (2x^3 \ddot{y} - 2x^2 \dot{y}) \partial x + (-4y^2 + 2x^2 y \ddot{y}) \partial y + (-2x^2 \dot{y} \ddot{y} - 8y \dot{y} + 4xy \ddot{y} \\ + 4x \dot{y}^2) \partial \dot{y} + (-6x^2 \ddot{y}^2 - 4y \ddot{y} - 12x \dot{y} \ddot{y} - 4 \dot{y}^2) \partial \ddot{y}$$

$$[X_3, X_8] = (3x^2)\partial_x + (-3x^2\dot{y} + 2x^3\ddot{y} + 8xy)\partial_y + (-4x\dot{y} + 8y + 3x^2\ddot{y})\partial_{\dot{y}} + (-4x\ddot{y} + 4\dot{y})\partial_{\ddot{y}}$$

$$[X_3, X_9] = (x)\partial_x + (-x\dot{y} + x^2\ddot{y})\partial_y + (-2\dot{y} + x\ddot{y})\partial_{\dot{y}} + (-2\ddot{y})\partial_{\ddot{y}}$$

$$[X_3, X_{10}] = \partial_x + (-\dot{y})\partial_y + (-\ddot{y})\partial_{\dot{y}}$$

$$[X_3, X_{11}] = (-x\ddot{y})\partial_y + (-\ddot{y})\partial_{\dot{y}}$$

$$[X_3, X_{12}] = (-\ddot{y})\partial_y$$

$$[X_3, X_{15}] = (6x^2)\partial_x + (10xy + x^3\ddot{y})\partial_y + (-2x\dot{y} + 10y + 3x^2\ddot{y})\partial_{\dot{y}} + (-8x\ddot{y} + 8\dot{y})\partial_{\ddot{y}}$$

$$[X_3, X_{14}] = (2x)\partial_x + (-2y + x^2\ddot{y})\partial_y + (-4\dot{y} + 2x\ddot{y})\partial_{\dot{y}} + (-4\ddot{y})\partial_{\ddot{y}}$$

$$[X_8, X_9] = (-8x^2\dot{y}^2 + 12x^3\dot{y}\ddot{y} - 4x^4\ddot{y}^2 + 8xy\dot{y} - 8x^2y\ddot{y})\partial_x + (-12x^2\dot{y}^2\ddot{y} + 4x^3\dot{y}\ddot{y}^2 + 8x\dot{y}^3 + 8xy\dot{y}\ddot{y} - 8\dot{y}^2y)\partial_{\dot{y}} + (-24x^2\dot{y}\ddot{y}^2 + 8x^3\ddot{y}^3 + 16x\dot{y}^2\ddot{y} + 16xy\dot{y}\ddot{y} - 16y\dot{y}\ddot{y})\partial_{\ddot{y}}$$

$$[X_8, X_{10}] = (-12x\dot{y}^2 + 12x^2\dot{y}\ddot{y} - 4x^3\ddot{y}^2 + 8y\dot{y})\partial_x + (4\dot{y}^3 - 8y\dot{y}\ddot{y})\partial_{\dot{y}} + (8\dot{y}^2\ddot{y} - 16y\dot{y}\ddot{y})\partial_{\ddot{y}}$$

$$[X_8, X_{11}] = (-12x^2\dot{y}\ddot{y} + 6x^3\ddot{y}^2 + 2x\dot{y}^2 + 8xy\dot{y})\partial_x + (12x\dot{y}^2\ddot{y} - 6x^2\dot{y}\ddot{y}^2 - 8y\dot{y}\ddot{y} - 2\dot{y}^3)\partial_{\dot{y}} + (24x\dot{y}\ddot{y}^2 - 12x^2\ddot{y}^3 - 4\dot{y}^2\ddot{y} - 16y\dot{y}\ddot{y})\partial_{\ddot{y}}$$

$$[X_8, X_{12}] = (-18x\dot{y}\ddot{y} + 12x^2\ddot{y}^2 + 2\dot{y}^2 + 8y\dot{y})\partial_x + (6\dot{y}^2\ddot{y} - 6x\dot{y}\ddot{y}^2)\partial_{\dot{y}} + (12\dot{y}\ddot{y}^2 - 12x\dot{y}^3)\partial_{\ddot{y}}$$

$$[X_8, X_{13}] = (-48x^2y\dot{y} + 24x^3y\ddot{y} - 24x^4\ddot{y}^2 + 6x^5\ddot{y}^2 + 24x^3\dot{y}^2 + 24xy^2)\partial_x + (-24x^2\dot{y}^3 + 24x^3\dot{y}^2 - 6x^4\dot{y}\ddot{y}^2 - 24x^2y\dot{y}\ddot{y} + 48xy\dot{y}^2 - 24y^2\dot{y})\partial_{\dot{y}} + (-48x^2\dot{y}^2\ddot{y} + 48x^3\dot{y}\ddot{y}^2 - 48x^2y\ddot{y}^2 - 12x^4\ddot{y}^3 + 96xy\dot{y}\ddot{y} - 48y^2\ddot{y})\partial_{\ddot{y}}$$

$$[X_8, X_{14}] = (-2x^3\dot{y}\ddot{y} + 4x^2\dot{y}^2 - 12xy\dot{y} + 8y^2 + 4x^2y\ddot{y})\partial_x + (-6x^2\dot{y}^2\ddot{y} + 2x^3\dot{y}\ddot{y}^2 - 4y\dot{y}^2 + 4xy\dot{y}\ddot{y} + 4x\dot{y}^3)\partial_{\dot{y}} + (-12x^2\dot{y}\ddot{y}^2 + 4x^3\ddot{y}^3 - 8y\dot{y}\ddot{y} + 8xy\dot{y}\ddot{y} + 8x\dot{y}^2\ddot{y})\partial_{\ddot{y}}$$

$$[X_9, X_{10}] = (-2\dot{y}^2 + 4x\dot{y}\ddot{y} - 2x^2\ddot{y}^2)\partial_x$$

$$[X_9, X_{11}] = (-x\dot{y}\ddot{y} + x^2\ddot{y}^2)\partial_x + (-x\dot{y}\ddot{y}^2 + \dot{y}^2\ddot{y})\partial_{\dot{y}} + (-2x\dot{y}^3 + 2\dot{y}\ddot{y}^2)\partial_{\ddot{y}}$$

$$[X_9, X_{12}] = (-3\dot{y}\ddot{y} + 4x\ddot{y}^2)\partial_x + (-\dot{y}\ddot{y}^2)\partial_{\dot{y}} + (-2\dot{y}^3)\partial_{\ddot{y}}$$

$$[X_9, X_{13}] = (-10xy\dot{y} + 10x^2y\ddot{y} - 15x^3\dot{y}\ddot{y} + 5x^4\ddot{y}^2 + 10x^2\dot{y}^2)\partial_x + (-10xy\dot{y}\ddot{y})$$

$$-5x^3\dot{y}\ddot{y}^2-10x\dot{y}^3+15x^2\dot{y}^2\ddot{y}+10y\dot{y}^2)\partial\dot{y}+(-20xy\dot{y}^2-10x^3\ddot{y}^3-20x\dot{y}^2\ddot{y}+30x^2\dot{y}\ddot{y}^2+20y\dot{y}\ddot{y})\partial\ddot{y}$$

$$[X_9, X_{14}] = (-5x^2\dot{y}\ddot{y}+2x^3\ddot{y}^2-2y\dot{y}+2x\dot{y}^2+4xy\ddot{y})\partial x + (2x\dot{y}^2\ddot{y}-x^2\dot{y}\ddot{y}^2-2y\dot{y}\ddot{y})\partial\dot{y} + (4x\dot{y}\ddot{y}^2-2x^2\ddot{y}^3-4y\dot{y}^2)\partial\ddot{y}$$

$$[X_{10}, X_{11}] = (2\dot{y}\ddot{y}-2x\dot{y}^2)\partial x$$

$$[X_{10}, X_{12}] = (-2\ddot{y}^2)\partial x$$

$$[X_{10}, X_{13}] = (-4y\dot{y}-6x^2\dot{y}\ddot{y}+2x^3\ddot{y}^2+6x\dot{y}^2)\partial x + (-2\dot{y}^3+4y\dot{y}\ddot{y})\partial\dot{y} + (-4\dot{y}^2\ddot{y}+8y\ddot{y}^2)\partial\ddot{y}$$

$$[X_{10}, X_{14}] = (2\dot{y}^2-4x\dot{y}\ddot{y}+2x^2\ddot{y}^2)\partial x$$

$$[X_{11}, X_{12}] = (-3\ddot{y}^2)\partial x$$

$$[X_{11}, X_{13}] = (-4xy\ddot{y}-6x^3\ddot{y}^2+12x^2\dot{y}\ddot{y}-4x\dot{y}^2)\partial x + (-12x\dot{y}^2\ddot{y}+6x^2\dot{y}\ddot{y}^2+4y\dot{y}\ddot{y}+4\dot{y}^3)\partial\dot{y} + (-24x\dot{y}\ddot{y}^2+12x^2\ddot{y}^3+8\dot{y}^2\ddot{y}+8y\ddot{y}^2)\partial\ddot{y}$$

$$[X_{11}, X_{14}] = (-3x^2\ddot{y}^2-2y\ddot{y}+4x\dot{y}\ddot{y})\partial x + (2x\dot{y}\ddot{y}^2-2\dot{y}^2\ddot{y})\partial\dot{y} + (4x\ddot{y}^3-4\dot{y}\ddot{y}^2)\partial\ddot{y}$$

$$[X_{12}, X_{13}] = (2y\ddot{y}-9x^2\ddot{y}^2-4\dot{y}^2+12x\dot{y}\ddot{y})\partial x + (-6\dot{y}^2\ddot{y}+6x\dot{y}\ddot{y}^2)\partial\dot{y} + (-12\dot{y}\ddot{y}^2+12x\ddot{y}^3)\partial\ddot{y}$$

$$[X_{12}, X_{14}] = (-6x\ddot{y}^2+4\dot{y}\ddot{y})\partial x + (2\dot{y}\ddot{y}^2)\partial\dot{y} + (4\ddot{y}^3)\partial\ddot{y}$$

$$[X_{13}, X_{14}] = (-4x^2y\ddot{y}-3x^4\ddot{y}^2+4y^2-4x^2\dot{y}^2+8x^3\ddot{y})\partial x + (8xy\dot{y}\ddot{y}+4x^3\dot{y}\ddot{y}^2-8y\dot{y}^2-12x^2\dot{y}^2\ddot{y}+8x\dot{y}^3)\partial\dot{y} + (16xy\ddot{y}^2+8x^3\ddot{y}^3-16y\dot{y}\ddot{y}^2-24x^2\dot{y}\ddot{y}^2+16x\dot{y}^2\ddot{y})\partial\ddot{y}$$